

Breakdown of universality in multi-cut matrix models

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Abstract

We solve the puzzle of the disagreement between orthogonal polynomials methods and mean field calculations for random $N \times N$ matrices with a disconnected eigenvalue support. We show that the difference does not stem from a \mathbb{Z}_2 symmetry breaking, but from the discreteness of the number of eigenvalues. This leads to additional terms (quasiperiodic in N) which must be added to the naive mean field expressions. Our result invalidates the existence of a smooth topological large N expansion and some postulated universality properties of correlators. We derive the large N expansion of the free energy for the general 2-cut case. From it we rederive by a direct and easy mean-field-like method the 2-point correlators and the asymptotic orthogonal polynomials. We extend our results to any number of cuts and to non-real potentials.

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1 Introduction

Random Matrix Models have been introduced in order to give an approximate statistical description of quantum systems involving disorder, chaos, complexity or whatever prevents from solving the equations of motion exactly. Those models are described by a matrix (Hamiltonian, transfer matrix, or scattering matrix) of large size N , which is too complicated to be diagonalized exactly, and for which only statistical observations of the spectrum are available (see [1, 2] for a review on RMT).

Most of the quantities of interest and observables are related to the short range (in energy scale) behavior of the spectrum (indeed small energies correspond to long time evolution, i.e. to equilibrium thermodynamical properties). This is why the short range correlation functions are the most studied.

At first, the simplest models assumed a gaussian weight for the random matrix and gave good agreement with observations, provided that the ensemble of matrices (hermitian, orthogonal, quaternionic...) has the required symmetries (time reversibility,...) [2, 3].

It has been observed that the correlation functions of the spectrum possess universal properties at short range, which do not depend on the probability weight, gaussian or not. This universality has been proved for a wide range of models by several approaches [3, 4, 5, 6, 7], but a very general proof and the exact hypothesis which lead to it are still under investigations.

Moreover, the long range correlation functions appear to share also some universal properties, which depend on the probability weight only through a few parameters [8, 7]. The most striking example is the 2-point correlation function that we shall discuss below.

In the following we will restrict our attention to the so called Hermitian-One-Matrix-Model⁵ [2, 9] (hermiticity corresponds to a system with broken time reversibility, for instance in the presence of a magnetic field).

We consider a hermitian matrix M of size $N \times N$ with a probability law of the form:

$$\mathcal{P}(M) = e^{-N \operatorname{tr} V(M)}$$

where V is a polynomial potential bounded from below.

We wish to study the statistical properties of the eigenvalues $(\lambda_1, \dots, \lambda_N)$ of M in the large N limit, in particular the density of eigenvalues $\rho(\lambda)$, and the correlation function $R(\lambda, \mu)$, which measures the probability that two of the eigenvalues take the values λ and μ .

⁵The other ensembles are of course worth considering, but this one is the simplest.

Roughly speaking, the eigenvalues tend to occupy a finite interval centered around the bottom of the potential well, and in the large N limit, the density of eigenvalues $\rho(\lambda)$ is a continuous function with a compact support.

The simplest case, where the support is connected, known as the “1-cut case”, has been extensively studied [2]. It is then found that the density $\rho(\lambda)$ is not universal, it depends on the details of the potential V , while the connected correlation function $R_c(\lambda, \mu)$ is universal in the short range regime ($|\lambda - \mu| \sim O(1/N)$), but also in the long range regime ($|\lambda - \mu| \sim O(1)$) once the short range oscillations (of period $\sim O(1/N)$) have been smoothed out.

What happens when the potential V possesses several wells, of approximately the same depths? Then the density $\rho(\lambda)$ has a disconnected support, $[a_1, b_1] \cup [a_2, b_2] \cup \dots \cup [a_s, b_s]$, each interval $[a_i, b_i]$ being centered around one well of the potential V . This case is known as the multicut (or multiband) case (here s cuts).

In the multicut case the density is still not universal, whereas the 2-point correlation function is universal in the short range regime and seems to have some universal properties in the long range regime after smoothing: in [10] an explicit form of the 2-point connected correlation function was given, and is claimed to be universal: indeed, according to the authors of [10] “it depends only on the number of connected components of the support and on the position of the endpoints, but not on the potential”. However, more recently several authors [11, 12, 13] have studied the two-cut case $s = 2$: they concentrated on the case of an even potential V (the two cuts are thus symmetric $[a, b]$ and $[-b, -a]$). Using an ansatz for the asymptotic expression of orthogonal polynomials in the large N limit, and rederiving the two-point function from this ansatz, they observed that the connected correlation function is still universal in the short distance regime (which was expected), but more surprisingly, that the smoothed connected correlation function in the long range regime depends on the parity of N (N being the size of the matrix). This seems to contradict the former result of [10] !

In this paper, we will solve this paradox.

We will show that the semi-classical method of [10] gives the 2-point connected correlation function only up to an additional non-universal term, which is already present in the free energy, but subdominant at large N in this case. We correct the semi-classical argument of [10], and give a simple (and physically appealing) derivation of the origin of the additional term, that we compute explicitly. This allows us to recover the results of [11, 13] for the symmetric case, and to generalize them to non symmetric potentials, without using orthogonal polynomials.

Using the same semi-classical argument, we are able to derive large N asymptotics for the orthogonal polynomials, recovering the results of [11, 13] as well as the general s cuts asymptotics which appeared recently in the mathematical literature [14], and to extend these results to the case of complex potentials.

The effect leading to the new term in the semi-classical calculation is simple enough to be explained briefly in this introductory section.

In [10], the free energy F of the matrix model is derived by a saddle point approximation. In particular, one has to extremize the action with respect to variations of the number n_i ($i = 1 \dots s$) of eigenvalues in each connected part of the support, or in other words with respect to the occupation ratio $x_i = n_i/N$:

$$F = F(x_c) \quad \text{where} \quad \left. \frac{\partial F}{\partial x} \right|_{x=x_c} = 0 \quad (1.1)$$

However, one has here missed the crucial fact that $n_i = Nx_i$ are not real numbers but integers. When Nx_c is not an integer, the extremum of $F(x)$ is never reached, and the saddle point approximation has to be slightly modified. Roughly speaking the discrete sum cannot be approximated by an integral:

$$\sum_n e^{-g(n-Nx_c)^2} \neq \int dx e^{-N^2 g(x-x_c)^2} \quad (1.2)$$

The discrete sum actually depends on how far from an integer Nx_c is. For instance, in the symmetric case, we have $x_c = \frac{1}{2}$, and the result depends on the parity of N .

We will show that (as expected from Eq. (1.2)) in general the result involves elliptic theta functions depending on Nx_c , thus leading to a quasi-periodic dependence on N . This effect is of order N^{-2} for the free energy, but is of order 1 for the computation of the orthogonal polynomials and for the correlation functions. It implies in particular that there is **no regular large N topological expansion** (involving only power series in N^{-2}) for the 2-cut matrix model.

We will find out that the short range correlation function is universal, while the long range smoothed correlation depends on N quasi-periodically.

The paper is divided as follows: in section 2 we introduce the method and notations for the 2-cut model, and we compute the free energy. In section 3 we derive the 2-point correlation function, and we recover the expression of [11, 13, 14] in the symmetric case. In section 4, we give an asymptotic expression for the orthogonal polynomials, which we use to rederive the universal short range properties of the spectrum, as well as the smoothed long range 2-point correlation function.

The generalizations to a complex potential or to an arbitrary number of cuts are presented in Appendix B and C. Appendix A is a summary of some relationships between elliptical functions in case the reader is not familiar with them.

2 The free energy

2.1 Basics

We start from the standard Hermitian matrix model defined by the partition function

$$Z[V; N] = \int d_N[M] e^{-N \operatorname{tr} V(M)} \quad (2.1)$$

where N is the dimension of the matrix M , V is an analytic – in general polynomial – and for the moment real – function, and $d_N[M]$ is the standard $U(N)$ invariant measure over Hermitian matrices

$$d_N[M] = \prod_{i=1}^N dM_{ii} \prod_{1 \leq i < j \leq N} 2 d\operatorname{Re}(M_{ij}) d\operatorname{Im}(M_{ij}) \quad (2.2)$$

Integrating out the "angular part" of M , Z can be rewritten as an integral over the N eigenvalues $\lambda_1, \dots, \lambda_N$ of M [2]

$$Z[V; N] = \mathbf{C}_N \tilde{Z}[V; N] \quad (2.3)$$

$$\tilde{Z}[V; N] = \int \prod_{k=1}^N d\lambda_k e^{-N \sum_k V(\lambda_k)} \prod_{k < l} (\lambda_k - \lambda_l)^2 = \int \prod_{k=1}^N d\lambda_k e^{-N^2 S(\lambda_1, \dots, \lambda_N)} \quad (2.4)$$

with the measure factor

$$\mathbf{C}_N = \operatorname{Vol} \left[\frac{\mathbf{U}(N)}{\mathbf{U}(1)^N \times \mathfrak{S}_N} \right] = \frac{1}{N!} \prod_{K=1}^N \frac{(2\pi)^{K-1}}{\Gamma(K)} \quad (2.5)$$

and with the action $S(\lambda_k)$:

$$S(\lambda_1, \dots, \lambda_N) = \frac{1}{N} \sum_{k=1}^N V(\lambda_k) - \frac{1}{2N^2} \sum_{1 \leq k \neq l \leq N} \ln(\lambda_k - \lambda_l)^2 \quad (2.6)$$

In the simplest "one cut" case, corresponding in particular to a concave potential, it is known [9] that the free energy F defined as ⁶

$$Z[V; N] = \left(\frac{2\pi}{N} \right)^{\frac{N^2}{2}} e^{-F[V; N]} \quad (2.7)$$

has a topological large N expansion

$$F = N^2 F_0 + F_1 + N^{-2} F_2 + \dots \quad (2.8)$$

⁶ Z is normalized here so that F is zero for the Gaussian model $V = \frac{1}{2}M^2$

obtained for instance by re-organizing the perturbative expansion according to the topology of the Feynman diagrams. The large N limit (planar limit) can be described by a "master field" configuration where the eigenvalues are described by a continuous density $\rho(\lambda)$ with a connected compact support $\mathcal{C} = [a, b]$, with the constraints

$$\int_{\mathcal{C}} d\lambda \rho(\lambda) = 1 \quad \text{and} \quad \rho(\lambda) \geq 0 \text{ if } \lambda \in \mathcal{C} \quad (2.9)$$

and the action 2.6 becomes

$$S[\rho] = \int_{\mathcal{C}} d\lambda V(\lambda) \rho(\lambda) - \int_{\mathcal{C} \times \mathcal{C}} d\lambda d\mu \rho(\lambda) \rho(\mu) \ln |\lambda - \mu| \quad (2.10)$$

The leading term of the free energy F_0 can be obtained by the saddle point method: the effective action $S[\rho]$ is extremized for a continuous distribution ρ_c and we have simply

$$F_0 = S[\rho_c] \quad (2.11)$$

(up to an additive – potential independent – constant). To compute ρ_c we include the constraint 2.9 in the effective action by a Lagrange multiplier Γ

$$\bar{S}[\rho] = S[\rho] + \Gamma(1 - \int_{\mathcal{C}} \rho) \quad (2.12)$$

The saddle point equation for ρ reads:

$$\frac{\partial \bar{S}}{\partial \rho(\lambda)} = V(\lambda) - 2 \int_{\mathcal{C}} d\mu \rho(\mu) \ln(|\lambda - \mu|) - \Gamma = 0 \quad \forall \lambda \in \mathcal{C} \quad (2.13)$$

which simply means that the real part of the effective potential

$$V_{\text{eff}}(\lambda) = V(\lambda) - 2 \int_{\mathcal{C}} d\mu \rho(\mu) \ln(\lambda - \mu) \quad (2.14)$$

is constant on the e.v. support \mathcal{C} , and equal to Γ . The derivative of 2.13 w.r.t. λ gives the well-known equation

$$\text{Re}(\omega_0(\lambda)) = \int_{\mathcal{C}} d\mu \rho(\mu) \frac{1}{\lambda - \mu} = V'(\lambda)/2 \quad \forall \lambda \in \mathcal{C} \quad (2.15)$$

where ω_0 is the large N resolvent

$$\omega_0(\lambda) = \lim_{N \rightarrow \infty} \left\langle \frac{1}{N} \text{Tr} \left[\frac{1}{\lambda - M} \right] \right\rangle = \int_{\mathcal{C}} d\mu \frac{\rho(\mu)}{\lambda - \mu} \quad (2.16)$$

Finally let us recall that in the one-cut case $\mathcal{C} = [a, b]$, if the potential V is a polynomial of degree P , ω is of the form

$$\omega_0(\lambda) = \frac{V'(\lambda)}{2} - \frac{M(\lambda) \sqrt{\sigma(\lambda)}}{2} \quad \text{with} \quad \sigma(\lambda) = (\lambda - a)(\lambda - b) \quad (2.17)$$

where $M(\lambda)$ is a polynomial with degree $P - 2$. a, b and M are entirely determined by the constraint that

$$\omega_0(\lambda) = \lambda^{-1} + \mathcal{O}(\lambda^{-2}) \quad \text{for} \quad \lambda \rightarrow \infty \quad (2.18)$$

The e.v. density is given by the discontinuity of ω

$$\rho(\lambda) = \frac{i}{2\pi} [\omega(\lambda + i0_+) - \omega(\lambda - i0_+)] = \frac{M(\lambda)\sqrt{|\sigma(\lambda)|}}{2\pi} \quad (2.19)$$

2.2 The 2-cut case

2.2.1 Mean field:

If the potential V is real but has more than one minimum, the large N limit may be described by an e.v. distribution on several disconnected intervals. For simplicity we shall first consider the case where there are two intervals

$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \quad , \quad \mathcal{C}_1 = [a, b] \quad , \quad \mathcal{C}_2 = [c, d] \quad , \quad a < b < c < d \quad (2.20)$$

In this case, as we shall see, **there is no topological large N expansion**, even for the free energy F . As shown in [15, 16], to describe the large N limit, we have to consider as an additional variable the "average" proportion of eigenvalues $x_1 = n_1/N$ and $x_2 = n_2/N$ in each interval \mathcal{C}_1 and \mathcal{C}_2 , and introduce the associated Lagrange multipliers Γ_1 and Γ_2 for the constraints

$$x_\alpha = \int_{\mathcal{C}_\alpha} \rho(\lambda) d\lambda \quad , \quad \alpha = 1, 2 \quad (2.21)$$

The effective action 2.12 now reads, with

$$x = x_1 \quad (2.22)$$

$$\bar{S}[\rho; x] = S[\rho] + \sum_{\alpha=1}^2 \Gamma_\alpha (x_\alpha - \int_{\mathcal{C}_\alpha} \rho(\lambda) d\lambda) \quad , \quad x_1 + x_2 = 1 \quad (2.23)$$

with $S[\rho]$ given by 2.10 as before. The saddle point equation w.r.t. $\rho(\lambda)$ gives as before the equation 2.13, which implies that the effective potential defined by 2.14 is constant on each interval

$$V_{\text{eff}}(\lambda) = \Gamma_\alpha \quad \text{when} \quad \lambda \in \mathcal{C}_\alpha \quad (2.24)$$

but the corresponding e.v. density $\rho_c(\lambda)$ and the effective action \bar{S}_c still depend explicitly of the e.v. proportion x , since we have

$$\bar{S}_c[x] = \frac{1}{2} \left(\int_{\mathcal{C}} \rho_c(\lambda) V(\lambda) d\lambda + \sum_{\alpha} \Gamma_\alpha x_\alpha \right) \quad (2.25)$$

The saddle point equation w.r.t. x implies the equality of the effective potentials for each interval

$$\frac{\partial \bar{S}}{\partial x} = \Gamma_1 - \Gamma_2 = 0 \quad (2.26)$$

This fixes the value of x , and it is known that with this last equation the e.v. density ρ_c is uniquely determined in the 2-cut case. The large N free energy is then given simply by

$$F_0 = \bar{S}[\rho_c; x_c] = \bar{S}_c[x_c] \quad (2.27)$$

For an explicit polynomial potential V of degree P , and for fixed x , the 2-cut mean field solution for the resolvent is

$$\omega_0(\lambda, x) = \frac{V'(\lambda)}{2} - \frac{M(\lambda)\sqrt{\sigma(\lambda)}}{2} \quad \text{with} \quad \sigma(\lambda) = (\lambda - a)(\lambda - b)(\lambda - c)(\lambda - d) \quad (2.28)$$

and $M(\lambda)$ a polynomial with degree $P - 3$. The e.v. density $\rho(\lambda, x)$ is still given by the discontinuity of ω_0 . The coefficients of M and the 4 end-points a, b, c, d are entirely determined by the constraint that $\omega_0(\lambda) \simeq \lambda^{-1}$ when $\lambda \rightarrow \infty$ and by the fact that x must be given by

$$x = \int_a^b \rho(\lambda, x) d\lambda = \frac{1}{2\pi} \int_a^b |M(\lambda, x)| \sqrt{|\sigma(\lambda)|} d\lambda \quad (2.29)$$

Finally the equation 2.26 which fixes $x = x_c$ reads

$$0 = V_{\text{eff}}(b) - V_{\text{eff}}(c) = \int_b^c d\lambda (2\omega_0(\lambda, x) - V'(\lambda)) = - \int_b^c d\lambda M(\lambda, x) \sqrt{|\sigma(\lambda)|} \quad (2.30)$$

2.2.2 Discreteness of number of e.v.'s:

This is sufficient if one is interested in the leading term in the large N limit (planar approximation). However, in order to understand the structure of the subdominant terms of the large N expansion, it turns out that we cannot neglect the fact that the number of e.v. $n_\alpha = Nx_\alpha$ in each interval \mathcal{C}_α must be an integer.

Let us consider the simple case where the potential V has two separate minima z_1 and z_2 . What has to be done is first to fix the number of eigenvalues $n_1 = n$ (resp. $n_2 = N - n$) in the vicinity of z_1 (resp. z_2) in the partition function 2.5 by writing

$$\tilde{Z}[V; N] = \sum_{n=0}^N \frac{N!}{n!(N-n)!} \tilde{Z}[V; n, N-n] \quad (2.31)$$

where

$$\tilde{Z}[V; n, N-n] = \int_{-\infty}^E \prod_{i \leq n} d\lambda_i \int_E^{+\infty} \prod_{j > n} d\lambda_j e^{-N \sum_k V(\lambda_k)} \prod_{k < l} (\lambda_k - \lambda_l)^2 \quad (2.32)$$

with E a "frontier" $b < E < c$ between the two semi-classical cuts $[a, b]$ and $[c, d]$. We now claim that each term of this discrete sum has a well defined large N topological expansion. Indeed, we can rewrite 2.32 as a matrix integral over two separate matrices: a $n_1 \times n_1$ matrix M_1 with the $n_1 = n$ e.v. $< E$ and a $n_2 \times n_2$ matrix M_2 with the $n_2 = N - n$ e.v. $> E$, as

$$\tilde{Z}[V; n] = \frac{1}{C_n C_{N-n}} \int d_{n_1}[M_1] \int d_{n_2}[M_2] e^{-N \text{Tr}(V(M_1)) - N \text{Tr}(V(M_2)) + 2 \text{Tr}(\ln(M_1 \otimes \text{Id} + \text{Id} \otimes M_2))} \quad (2.33)$$

This last matrix integral has a topological large N expansion of the form 2.8 in the 't Hooft limit $N \rightarrow \infty$, $x = n/N$ fixed, obtained by doing a classical perturbative expansion around the smallest minimum z_1 of V for M_1 and around the largest minimum z_2 of V for M_2 , and by re-organizing the perturbative expansion according to the topology of the Feynman diagrams. Taking into account carefully the measure factors C_n and C_{N-n} , and using their large N asymptotics

$$C_N = \frac{1}{N!} \left(\frac{2\pi}{N} \right)^{\frac{N^2}{2}} e^{\frac{3}{4}N^2} (2\pi)^{-N} N^{\frac{1}{12}} \text{cst} (1 + \mathcal{O}(N^{-1})) \text{ when } N \rightarrow \infty \quad (2.34)$$

(easily derived from Stirling formula), we obtain that

$$Z[V; N] = \left(\frac{2\pi}{N} \right)^{\frac{N^2}{2}} N^{-\frac{1}{12}} \sum_{n=0}^N e^{-F[V; N, x]} \quad (2.35)$$

where each $F[V; N, x]$ has a regular large N asymptotic expansion of the form

$$F[V; N, x] = \sum_{h=0}^{\infty} N^{2-2h} F_h[V, x] \quad \text{where} \quad x = n/N \quad (2.36)$$

with each $F_h[V, x]$ a regular function of $x = n/N$. In particular, the leading large N term is given (up to an additive $-V$ **and** x independent $-$ constant) by the classical effective action 2.27

$$F_0 = \bar{S}[\rho_c; x_c] = \bar{S}_c[x_c] \quad (2.37)$$

Finally, let us stress that although this decomposition depends on the arbitrary parameter E , since E is in the interval $]b, c[$ where the density of eigenvalues is exponentially small with N , the integral 2.32 depends on E only through exponentially small terms of order $e^{-\text{cst} \cdot N}$, which are "non-perturbative" in the topological expansion Eq. (2.36).

2.2.3 Beyond mean-field:

We can now easily calculate the subleading terms of order $\mathcal{O}(N^{-2})$ for the full partition function. In the large N limit we can approximate the sum 2.36 by

$$Z[V; N] \propto \sum_{n=0}^N e^{-N^2 F_0[V; x] - F_1[V; x] + \dots} \quad (2.38)$$

If x_c denotes the saddle point of $F_0[x]$ given by 2.26, the sum is dominated by the n 's such that

$$|n - Nx_c| = \mathcal{O}(1) \quad (2.39)$$

Thus we can still use a quadratic approximation for $F_0[x]$

$$Z[V; N] \propto e^{-(N^2 F_0[V; x_c] + F_1[V; x_c] + \dots)} \sum_n e^{-(n - Nx_c)^2 F_0''[V; x_c]/2} \quad (2.40)$$

where $F_0'' = \partial^2 F_0 / \partial x^2$ and where the \dots represent terms of order $\mathcal{O}(N^{-2})$. The last sum over n gives simply an elliptic Jacobi theta function θ_3

$$\sum_n e^{-(n - Nx_c)^2 F_0''[V; x_c]/2} = (2\pi F_0''[V; x_c])^{-1/2} \theta_3(Nx_c | \tau) \quad (2.41)$$

with modular parameter τ given by

$$\tau = \frac{2i\pi}{F_0''[V; x_c]} \quad (2.42)$$

and where the theta function is defined as

$$\theta_3(z | \tau) = \theta_3(z) = \sum_{n \in \mathbb{Z}} q^{n^2} e^{2i\pi n z} \quad \text{with} \quad q = e^{i\pi\tau} \quad (2.43)$$

It obeys the periodicity relations

$$\theta_3(z + 1) = \theta_3(z) \quad , \quad \theta_3(z + \tau) = e^{-i\pi(2z + \tau)} \theta_3(z) \quad (2.44)$$

(For details on elliptic functions see e.g. Refs [24, 25, 23]). Eventually we have for the free energy

$$\begin{aligned} F[V; N] = & N^2 F_0[V, x_c] \\ & - \ln(\theta_3(Nx_c)) + F_1(V; x_c) + \frac{1}{2} \ln(2\pi F_0''[V; x_c]) \\ & + \mathcal{O}(N^{-2}) \end{aligned} \quad (2.45)$$

where F_1 is the torus contribution in the topological expansion of 2.33. The next terms of this expansion can be calculated along the same line.

Let us stress that this is not a topological expansion, since the second term $\ln(\theta_3(Nx_c))$, seemingly $\mathcal{O}(1)$ and contributing at the torus order, is **not regular** in N . Indeed, it is periodic in x_c with period $1/N$. When computing some observables or quantities of the matrix model, one must take derivatives of F w.r.t. some parameters of the potential V . Since the saddle point x_c depends implicitly on V , every derivative will give a factor N , and this term may become of the same order than the first term $N^2 F_0[x_c]$ given by the planar limit. Note that the last two terms depend on x and not on Nx , and they will remain subdominant once we take derivatives of F .

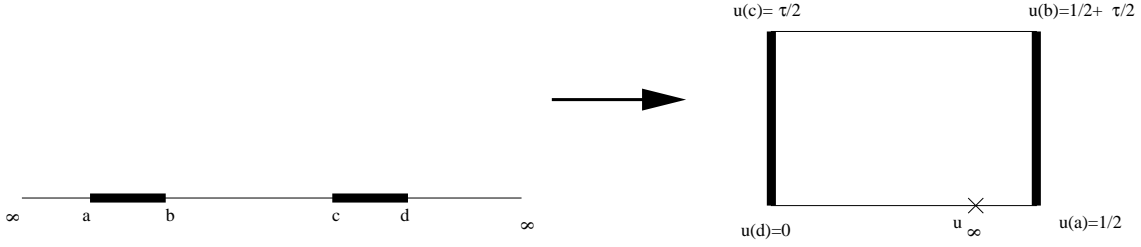


Figure 1: the upper half-plane is mapped onto a rectangle $(1/2, \tau/2)$

2.3 The modular parameter

Finally, we can express simply the modular parameter τ defined by Eq. (2.42) in term of the end-points a, b, c, d of the support of e.v. For this purpose, we introduce the function σ

$$\sigma(\lambda) = (\lambda - a)(\lambda - b)(\lambda - c)(\lambda - d) \quad (2.46)$$

and the function u

$$u(\lambda) = \frac{1}{2K} \int_d^\lambda \frac{dz}{\sqrt{\sigma(z)}} \quad (2.47)$$

where K is

$$K = \int_b^c \frac{dz}{\sqrt{|\sigma(z)|}} = \frac{2}{\sqrt{(c-a)(d-b)}} K[m] \quad \text{with} \quad m = \frac{(d-a)(c-b)}{(d-b)(c-a)} \quad (2.48)$$

$K[m]$ is the complete elliptic integral of the first kind. Similarly we define

$$K' = \int_a^b \frac{dz}{\sqrt{|\sigma(z)|}} = \frac{2}{\sqrt{(c-a)(d-b)}} K[m'] \quad \text{with} \quad m' = 1 - m \quad (2.49)$$

We shall show that the modular parameter τ of Eq. (2.42) coincides with the standard modular parameter of the torus associated to the mapping u , i.e. of the elliptic curve $y^2 = \sigma(z)$. Indeed, τ is simply given by

$$\tau = i \frac{K'}{K} = i \frac{K[1-m]}{K[m]} \quad (2.50)$$

So we have

$$u(d) = 0 \quad , \quad u(a) = \frac{1}{2} \quad , \quad u(b) = \frac{1+\tau}{2} \quad , \quad u(c) = \frac{\tau}{2} \quad , \quad u(\infty) = u_\infty \quad (2.51)$$

and u maps the upper half λ -plane onto the half-periods rectangle $(1/2, \tau/2)$ and the double-sheeted complex λ -plane onto the period rectangle $(1, \tau)$.

To show Eq. (2.50), we use the fact that in the two-cut case, if we fix x (the e.v. ratio in the first cut) the semiclassical e.v. density (extrema of the effective action \bar{S}) is now a function $\rho(\lambda, x)$ of λ and x , and the end-points a, b, c, d depend on x . Therefore the large N resolvent ω_0 is of the form

$$\omega_0(\lambda, x) = \frac{V'(\lambda)}{2} - \frac{M(\lambda, x)\sqrt{\sigma(\lambda)}}{2} \quad (2.52)$$

with $M(\lambda, x)$ a polynomial with degree $P - 3$ in λ (P being the degree of V), entirely fixed by the constraints 2.18 and 2.29. Therefore the partial derivative of $\omega_0(\lambda, x)$ w.r.t. x is necessarily of the form

$$\frac{\partial\omega_0(\lambda, x)}{\partial x} = \frac{C}{\sqrt{\sigma(\lambda)}} \quad (2.53)$$

with $C = C(\lambda, x)$ a priori a polynomial in λ . Since 2.18 still holds independently of x we must have

$$\frac{\partial\omega_0(\lambda, x)}{\partial x} = \mathcal{O}(\lambda^{-2}) \quad \text{for} \quad \lambda \rightarrow \infty \quad (2.54)$$

which implies that $C(\lambda, x)$ is of degree 0 in λ , i.e. is a constant (depending only on x)

$$C = C(x) \quad (2.55)$$

This constant can be easily determined by using that

$$x = \int_a^b \rho(\lambda, x) d\lambda = \int_{\mathcal{C}'} \frac{d\lambda}{2i\pi} \omega_0(\lambda, x) \quad (2.56)$$

with \mathcal{C}' a clockwise contour encircling the interval $[a, b]$. Therefore we have

$$\frac{\partial x}{\partial x} = 1 = \int_{\mathcal{C}'} \frac{d\lambda}{2i\pi} \frac{C}{\sqrt{\sigma(\lambda)}} = -\frac{CK'}{\pi} \Rightarrow C = -\frac{\pi}{K'} \quad (2.57)$$

with K' the half-period defined in Eq. (2.49). Now we use Eq. (2.26), Eq. (2.27) and the definition of the effective potential V_{eff} of Eq. (2.14) to write the derivative of the free energy w.r.t. x as

$$\frac{\partial F_0}{\partial x} = V_{\text{eff}}(b) - V_{\text{eff}}(c) = \int_b^c d\lambda (2\omega_0(\lambda) - V'(\lambda)) \quad (2.58)$$

Now we take the derivative w.r.t. x of this equation and obtain

$$F_0'' = \frac{\partial^2 F_0}{\partial x^2} = 2 \int_b^c d\lambda \frac{\partial\omega_0(\lambda)}{\partial x} = 2 \int_b^c d\lambda \frac{C}{\sqrt{\sigma(\lambda)}} = -2CK = \frac{2\pi K}{K'} \quad (2.59)$$

Using Eq. (2.42) we thus obtain the result 2.50.

3 2-points correlation function

3.1 The basic formula

As a first application we compute the large N smoothed connected two-point correlation function (first obtained by [11, 13, 14]), defined as

$$\omega^c(\lambda, \mu) = \langle \text{Tr} \left[\frac{1}{\lambda - M} \right] \text{Tr} \left[\frac{1}{\mu - M} \right] \rangle - \langle \text{Tr} \left[\frac{1}{\lambda - M} \right] \rangle \langle \text{Tr} \left[\frac{1}{\mu - M} \right] \rangle \quad (3.1)$$

Adding source terms to the potential of the form

$$V_{\epsilon_\lambda} = V(z) - \epsilon_\lambda \frac{1}{\lambda - z} \quad , \quad V_{\epsilon_\lambda, \epsilon_\mu}(z) = V(z) - \epsilon_\lambda \frac{1}{\lambda - z} - \epsilon_\mu \frac{1}{\mu - z} \quad (3.2)$$

we have

$$\omega^c(\lambda, \mu) = -\frac{1}{N^2} \frac{\partial}{\partial \epsilon_\lambda} \frac{\partial}{\partial \epsilon_\mu} F[V_{\epsilon_\lambda, \epsilon_\mu}, N] \Big|_{\epsilon_\lambda = \epsilon_\mu = 0} \quad (3.3)$$

and for the resolvent (one-point function)

$$\omega(\lambda) = \frac{1}{N} \langle \text{Tr} \left[\frac{1}{\lambda - M} \right] \rangle = -\frac{1}{N^2} \frac{\partial}{\partial \epsilon_\lambda} F[V_{\epsilon_\lambda}, N] \Big|_{\epsilon_\lambda = 0} \quad (3.4)$$

If the ϵ_λ 's are small and the λ 's not too close to the cuts, the mean-field solution is still a two-cut e.v. distribution, with $x_c = x_c(\epsilon)$ an explicit function of the λ 's. So from Eq. (2.45) for the free energy we have for the two-point function

$$\omega^c(\lambda, \mu) = \left[-\frac{\partial}{\partial \epsilon_\lambda} \frac{\partial}{\partial \epsilon_\mu} F_0[V_{\epsilon_\lambda, \epsilon_\mu}] + \frac{\partial x_c}{\partial \epsilon_\lambda} \frac{\partial x_c}{\partial \epsilon_\mu} [\ln(\theta_3(Nx_c))]'' \right]_{\epsilon_\lambda = \epsilon_\mu = 0} + \mathcal{O}(N^{-1}) \quad (3.5)$$

The first term in the r.h.s. of 3.5 is the mean-field contribution already calculated in [17, 10], the second term involving a second derivative of an elliptic function, characterizes the multi-cut solution.

3.2 The mean-field contribution

For completeness let us first rederive the mean field contribution of [17, 10]. Taking the derivative with respect to ϵ_λ we obtain the mean-field resolvent for the potential V_{ϵ_μ}

$$-\frac{\partial}{\partial \epsilon_\lambda} F_0[V_{\epsilon_\lambda, \epsilon_\mu}] \Big|_{\epsilon_\lambda = 0} = \omega_0(\lambda; V_{\epsilon_\mu}) \quad (3.6)$$

which must be of the form

$$\omega_0(z; V_{\epsilon_\mu}) = \frac{1}{2} \left[V'(z) - \frac{\epsilon_\mu}{(z - \mu)^2} + \frac{M(z)\sqrt{\sigma(z)}}{(z - \mu)^2} \right] \quad (3.7)$$

with $M(z)$ a polynomial of degree $P - 1$ (here both the coefficients of M and of σ depend on μ and ϵ_μ). In addition to the $P - 2$ constraints (coming from 2.18 and 2.30) $\omega_0(z; V_{\epsilon_\mu})$ must be regular at $z = \mu$. This determines entirely M . Taking the derivative w.r.t. ϵ_μ and using the symmetry $\lambda \leftrightarrow \mu$ we get

$$\omega_0^c(\lambda, \mu) = -\frac{\partial}{\partial \epsilon_\lambda} \frac{\partial}{\partial \epsilon_\mu} F_0 = -\frac{1}{2} \frac{1}{(\lambda - \mu)^2} \left[1 + \frac{Q(\lambda, \mu)}{\sqrt{\sigma(\lambda)}\sqrt{\sigma(\mu)}} \right] \quad (3.8)$$

with $Q(\lambda, \mu)$ a symmetric polynomial in λ and in μ . The constraints on ω_0^c are: (i) $\omega_0^c = \mathcal{O}(\lambda^{-2})$ as $\lambda \rightarrow \infty$ which implies that Q is of degree at most 2; (ii) ω_0^c is regular at $\lambda = \mu$ which implies that $Q(\lambda, \mu) = -\sigma((\lambda + \mu)/2) + \mathcal{O}((\lambda - \mu)^2)$; (iii) finally the equality of the effective potential on the two cuts implies that

$$\int_b^c \omega_0^c(\lambda, \mu) d\mu = 0 \quad (3.9)$$

Conditions (i) and (ii) fix uniquely Q

$$Q(\lambda, \mu) = -\frac{1}{2} \left[\begin{array}{l} (\lambda - a)(\mu - b)(\mu - c)(\lambda - d) \\ +(\mu - a)(\lambda - b)(\lambda - c)(\mu - d) \end{array} \right] + S(\lambda - \mu)^2 \quad (3.10)$$

up to a constant S fixed by condition (iii), which is found to be

$$S = -\frac{1}{2}(c - a)(d - b) \frac{E[m]}{K[m]} \quad \text{with} \quad m = \frac{(c - b)(d - a)}{(c - a)(d - b)} \quad (3.11)$$

and where $K[m]$ and $E[m]$ are the standard elliptic integrals of the first and second kind.

3.3 The non-regular contribution

In order to compute the second contribution to 3.5, we simply need $\frac{\partial x_c}{\partial \epsilon_\lambda}$. Since x_c is fixed by the constraint $\frac{\partial F_0}{\partial x} = 0$ we can write

$$\frac{\partial x_c}{\partial \epsilon_\lambda} = -\frac{\partial^2 F_0}{\partial x \partial \epsilon_\lambda} / \frac{\partial^2 F_0}{\partial x^2} \quad (3.12)$$

Using the results of subsection 2.3 we have

$$\frac{\partial^2 F_0}{\partial x^2} = \frac{2\pi K}{K'} \quad \text{and} \quad \frac{\partial^2 F_0}{\partial x \partial \epsilon_\lambda} = -\frac{\partial}{\partial x} \omega_0(\lambda, x) = \frac{\pi}{K'} \frac{1}{\sqrt{\sigma(\lambda)}} \quad (3.13)$$

So we have eventually

$$\frac{\partial x_c}{\partial \epsilon_\lambda} = -\frac{1}{2K\sqrt{\sigma(\lambda)}} \quad (3.14)$$

and the second non-regular term is

$$\frac{\partial x_c}{\partial \epsilon_\lambda} \frac{\partial x_c}{\partial \epsilon_\mu} [\ln(\theta_3(Nx_c|\tau))]'' = \frac{1}{2K\sqrt{\sigma(\lambda)}} \frac{1}{2K\sqrt{\sigma(\mu)}} [\ln(\theta_3(Nx_c|\tau))]'' \quad (3.15)$$

with K defined by Eq. (2.48). Using standard relations on elliptic functions, this can be rewritten as

$$\frac{(c-a)(d-b)}{4\sqrt{\sigma(\lambda)}\sqrt{\sigma(\mu)}} \left[-\frac{E[m]}{K[m]} + \operatorname{dn}^2(Nx_c + \frac{1}{2}) \right] \quad (3.16)$$

with

$$\operatorname{dn}(u) = \operatorname{dn}(2K[m]u|m) \quad (3.17)$$

where $\operatorname{dn}(u|m)$ is the Jacobi elliptic function dn . Its periods are $2K[m]$ and $4iK[m']$, and $\operatorname{dn}^2(z)$ has periods 1 and τ .

3.4 The final result

Combining 3.8, 3.10, 3.11 and 3.16 we obtain the final result for the 2-point correlation function

$$\begin{aligned} \omega^c(\lambda, \mu) &= -\frac{1}{4(\lambda - \mu)^2} \left[\left(1 - \sqrt{\frac{(\lambda - a)(\lambda - b)(\mu - c)(\mu - d)}{(\mu - a)(\mu - b)(\lambda - c)(\lambda - d)}} \right) + (\lambda \leftrightarrow \mu) \right] \\ &\quad - \frac{(c-a)(d-b)}{4\sqrt{\sigma(\lambda)}\sqrt{\sigma(\mu)}} \operatorname{sn}^2(Nx_c + \frac{1}{2}) \end{aligned} \quad (3.18)$$

We have used the relation $\operatorname{dn}^2(u) = 1 - m \operatorname{sn}^2(u)$, where similarly to 3.17 we note

$$\operatorname{sn}(u) = \operatorname{sn}(2K[m]u|m) \quad (3.19)$$

Surprisingly, the ratio $E[m]/K[m]$ characteristic of the mean-field solution of [17, 10] has disappeared.

The smoothed 2-point connected density correlator $\rho^c(\lambda, \mu)$, defined as

$$\rho^c(\lambda, \mu) = \langle \operatorname{Tr} [\delta(\lambda - M)] \operatorname{Tr} [\delta(\mu - M)] \rangle - \langle \operatorname{Tr} [\delta(\lambda - M)] \rangle \langle \operatorname{Tr} [\delta(\mu - M)] \rangle \quad (3.20)$$

can be obtained easily from the discontinuity of $\omega^c(\lambda, \mu)$. One obtains in the large N limit, if λ and μ are on the support of e.v.

$$\begin{aligned} \rho^c(\lambda, \mu) &= -\frac{1}{4\pi^2} \left[\frac{1}{(\lambda - \mu)^2} \left(\sqrt{\left| \frac{(\lambda - a)(\lambda - b)(\mu - c)(\mu - d)}{(\mu - a)(\mu - b)(\lambda - c)(\lambda - d)} \right|} + \lambda \leftrightarrow \mu \right) \right. \\ &\quad \left. + \varepsilon_\lambda \varepsilon_\mu \frac{(c-a)(d-b)}{\sqrt{|\sigma(\lambda)|}\sqrt{|\sigma(\mu)|}} \operatorname{sn}^2(Nx_c + \frac{1}{2}) \right] \end{aligned} \quad (3.21)$$

$$\varepsilon_\lambda = 1 \quad \text{if } \lambda \in [a, b] \quad , \quad -1 \quad \text{if } \lambda \in [c, d] \quad , \quad (3.22)$$

and zero otherwise.

The new non-regular term $\text{sn}^2(Nx_c + \frac{1}{2})$ is an even periodic function of Nx_c with period 1 which varies between 0 and 1. Therefore, as N varies, depending on the rationality or the irrationality of x_c , the two-point function will be varying with N in a periodic or quasiperiodic way.

3.5 The symmetric case

It is now very easy to recover the results of [11, 13] for a symmetric potential. Indeed, if the potential V is symmetric, the two cuts are also symmetric

$$a = -d \quad , \quad b = -c \quad (3.23)$$

and we have automatically

$$x_c = \frac{1}{2} \quad (3.24)$$

so that

$$\text{sn}^2(Nx_c + \frac{1}{2}) = \begin{cases} \text{sn}^2(\frac{1}{2}) = 1 & \text{if } N \text{ is even} \\ \text{sn}^2(0) = 0 & \text{if } N \text{ is odd} \end{cases} \quad (3.25)$$

Eq. (3.18) and Eq. (3.22) become

$$\omega^c(\lambda, \mu) = -\frac{1}{2(\lambda - \mu)^2} \left[1 - \frac{(a^2 - \lambda\mu)(b^2 - \lambda\mu)}{\sqrt{\sigma(\lambda)}\sqrt{\sigma(\mu)}} \right] - \frac{(-1)^N}{2} \frac{ab}{\sqrt{\sigma(\lambda)}\sqrt{\sigma(\mu)}} \quad (3.26)$$

$$\rho^c(\lambda, \mu) = \frac{1}{2\pi^2} \frac{\varepsilon_\lambda \varepsilon_\mu}{\sqrt{|\sigma(\lambda)|}\sqrt{|\sigma(\mu)|}} \left(\frac{(a^2 - \lambda\mu)(b^2 - \lambda\mu)}{(\lambda - \mu)^2} - (-1)^N ab \right) \quad (3.27)$$

with $\sigma(\lambda) = (\lambda^2 - a^2)(\lambda^2 - b^2)$.

3.6 The two-point function as an elliptic function

It is interesting to consider the two-point correlator in terms of the elliptic coordinates defined by Eq. (2.47)

$$u = u(\lambda) \quad , \quad v = u(\mu) \quad (3.28)$$

Let us thus consider

$$\bar{\omega}^c(u, v) = \frac{\partial \lambda}{\partial u} \frac{\partial \mu}{\partial v} \omega^c(\lambda, \mu) = 2K\sqrt{\sigma(\lambda)} 2K\sqrt{\sigma(\mu)} \omega^c(\lambda, \mu) \quad (3.29)$$

It is easy to see (from the properties of ω^c) that $\bar{\omega}^c(u, v)$ satisfies:

1. $\bar{\omega}^c(u, v)$ is a doubly periodic function of u (and of v) with periods 1 and τ ;

2. $\bar{\omega}^c(u, v)$ is regular at u and $v = u(a), u(b), u(c), u(d)$ and u_∞ ;
3. $\bar{\omega}^c(u, v)$ is regular when $u = v$, but has a double pole at $u = -v$ (corresponding to the double pole of $\omega^c(\lambda, \mu)$ when $\lambda = \mu$ but with λ in the first sheet and μ in the second sheet), with residue 1.

This implies that $\bar{\omega}^c(u, v)$ is a Weierstrass elliptic function

$$\bar{\omega}^c(u, v) = \wp(u + v|\tau) + \text{constant} \quad (3.30)$$

where the constant depends on Nx_c (\wp has periods 1 and τ). Using classical identities between the Weierstrass \wp function and the Jacobi elliptic functions, it can be easily calculated. We find the remarkably simple result

$$\bar{\omega}^c(u, v) = \wp(u + v|\tau) - \wp(Nx_c + \frac{\tau}{2}|\tau) \quad (3.31)$$

or equivalently

$$\bar{\omega}^c(u, v) = - [\ln(\theta_1(u + v|\tau))]'' + [\ln(\theta_3(Nx_c|\tau))]'' \quad (3.32)$$

4 The orthogonal polynomials

Let us briefly recall some basic facts about the well-known method of orthogonal polynomials [18], which is a powerful tool for studying the spectral properties of random matrices [2]. Asymptotic expressions for the orthogonal polynomials have been obtained recently [14] in the mathematical literature, by solving a Riemann-Hilbert problem. Here we will derive them from the free energy directly.

Consider the partition function (2.4):

$$\tilde{Z} = \int d\lambda_1 \dots d\lambda_N e^{-N \sum_i V(\lambda_i)} \prod_{i < j} (\lambda_i - \lambda_j)^2 \quad (4.1)$$

The last term is a Vandermonde determinant [2]:

$$\prod_{i < j} (\lambda_i - \lambda_j) = \det_{i,j} ((\lambda_i)^{j-1}) = \det_{i,j} (\mathcal{P}_{j-1}(\lambda_i)) \quad (4.2)$$

where the last equality is obtained by linearly mixing columns of the determinant, and holds for arbitrary monic polynomials $\mathcal{P}_n(\lambda)$ with leading coefficient $\mathcal{P}_n(\lambda) = \lambda^n + \dots$

The method of orthogonal polynomials consists in choosing a family of polynomials suitable for the computation of (4.1), namely, the family of polynomials orthogonal with respect to the weight $\exp -NV(\lambda)$:

$$\int d\lambda \mathcal{P}_n(\lambda) \mathcal{P}_m(\lambda) e^{-NV(\lambda)} = h_n \delta_{nm} \quad (4.3)$$

With this particular choice of polynomials, the integral (4.1) is merely:

$$\tilde{Z} = N! \prod_{n=0}^{N-1} h_n \quad (4.4)$$

and the joint probability density of all the eigenvalues takes the form of a Slater determinant:

$$R_N(\lambda_1, \dots, \lambda_N) = \frac{1}{N!} \left(\det_{\substack{0 \leq n < N \\ 1 \leq i \leq N}} [\psi_{n-1}(\lambda_i)] \right)^2 \quad (4.5)$$

where the wave functions $\psi_n(\lambda) = \frac{1}{\sqrt{h_n}} \mathcal{P}_n(\lambda) e^{-\frac{N}{2} V(\lambda)}$ are orthonormal.

4.1 The Kernel $K(\lambda, \mu)$

The square of a determinant can be rewritten as the determinant of a product:

$$\left(\det_{n,i} (\psi_{n-1}(\lambda_i)) \right)^2 = \det_{1 \leq i, j \leq N} \left[\sum_{n=0}^{N-1} \psi_n(\lambda_i) \psi_n(\lambda_j) \right]$$

we are thus led to introduce the kernel $K(\lambda, \mu)$ [19]:

$$K(\lambda, \mu) = \frac{1}{N} \sum_{n=0}^{N-1} \psi_n(\lambda) \psi_n(\mu) \quad (4.6)$$

In terms of which the joint density of eigenvalues is now a determinant

$$R_N(\lambda_1, \dots, \lambda_N) = \frac{N^N}{N!} \det [K(\lambda_i, \lambda_j)] \quad (4.7)$$

The orthonormality properties of the polynomials imply the projection relations

$$\int d\lambda K(\lambda, \lambda) = 1 \quad \text{and} \quad \int d\lambda K(\mu, \lambda) K(\lambda, \nu) = \frac{1}{N} K(\mu, \nu) \quad (4.8)$$

which make any partial integration of (4.7) easy to perform (theorem of Dyson [20]).

In particular, the integration over $N - 1$ eigenvalues gives the density of eigenvalues

$$\rho(\lambda_1) = \int d\lambda_2 \dots d\lambda_N R_N(\lambda_1, \dots, \lambda_N) = K(\lambda_1, \lambda_1)$$

and the integration over $N - 2$ eigenvalues gives the correlation function:

$$\begin{aligned} R_2(\lambda_1, \lambda_2) &= \int d\lambda_3 \dots d\lambda_N R_N(\lambda_1, \dots, \lambda_N) \\ &= \frac{N}{N-1} (K(\lambda_1, \lambda_1) K(\lambda_2, \lambda_2) - K(\lambda_1, \lambda_2) K(\lambda_2, \lambda_1)) \end{aligned}$$

In short:

$$\rho(\lambda) = K(\lambda, \lambda) \quad , \quad \rho(\lambda, \mu) = (K(\lambda, \lambda) K(\mu, \mu) - K(\lambda, \mu)^2) \quad (4.9)$$

In addition, the Darboux-Christoffel theorem [24, 21], asserts that

$$K(\lambda, \mu) = \frac{1}{Nh_{N-1}} \frac{\mathcal{P}_N(\lambda)\mathcal{P}_{N-1}(\mu) - \mathcal{P}_N(\mu)\mathcal{P}_{N-1}(\lambda)}{\lambda - \mu} e^{-\frac{N}{2}(V(\lambda)+V(\mu))} \quad (4.10)$$

which means that we need to evaluate \mathcal{P}_n only for $n = N$ and $n = N - 1$.

Thus, we shall now aim at finding asymptotic expressions for the orthogonal polynomials $\mathcal{P}_n(\lambda)$, and the kernel $K(\lambda, \mu)$ in the large N limit, and n close to N . This has been done in the 1-cut case [7] and in the symmetric 2-cut case [11, 14, 12]. Here we will generalize it to the non-symmetric case, with the method used in [21].

4.2 WKB approximation for the orthogonal polynomials $\mathcal{P}_n(\lambda)$

The orthogonal polynomials have the following integral representation (see Appendix 1 of [21] or [18]):

$$\mathcal{P}_n(\lambda) = \frac{\int dM_{n \times n} \det(\lambda - M) e^{-N \operatorname{tr} V(M)}}{\int dM_{n \times n} e^{-N \operatorname{tr} V(M)}} \quad (4.11)$$

where the integral is restricted to hermitian matrices of size $n \times n$.

Thus the orthogonal polynomial is given by the ratio of two matrix integrals of the same type as the partition function 2.1:

$$\mathcal{P}_n(\lambda) = \frac{Z[V + \delta V_1 + \delta V_2; n]}{Z[V + \delta V_1; n]} = \frac{e^{-F[V + \delta V_1 + \delta V_2; n]}}{e^{-F[V + \delta V_1; n]}} \quad (4.12)$$

where

$$\delta V_1(z) = \frac{N-n}{n} V(z) \quad \text{and} \quad \delta V_2 = -\frac{1}{n} \ln(z - \lambda) \quad (4.13)$$

We have seen in the previous section (eq.2.45) that

$$F[V; n] = n^2 F_0[V; x_c] - \ln \theta_3(n x_c[V]) + \dots \quad (4.14)$$

We will use the fact that under a variation δV of the potential, the variation of F_0 is [21, 8]:

$$\delta F_0 = \frac{1}{2i\pi} \oint \omega(z) \delta V(z) dz \quad (4.15)$$

where the anti-clockwise contour encloses the support of the density of eigenvalues, and $\omega(z)$ is the resolvent (eq 3.4 and 2.28):

$$\omega(z) = \frac{1}{2} \left(V'(z) - M(z) \sqrt{\sigma(z)} \right)$$

It is convenient to introduce two sources t_1 and t_2 for the variations δV_1 and δV_2 of the potential, and consider a generalized potential $\mathcal{V}(z)$:

$$\mathcal{V}(z) = V(z) + t_1 \delta V_1(z) + t_2 \delta V_2(z) = V(z) + t_1 V(z) + t_2 \ln |\lambda - z|$$

Since $t_1 = \frac{N-n}{n}$ and $t_2 = -\frac{1}{n}$ are both small of order $O(N^{-1})$, we will expand F in Taylor's series:

$$F[\mathcal{V}(z); n] = F[V(z); n] + t_1 \partial_1 F + t_2 \partial_2 F + \frac{t_1^2}{2} \partial_{11} F + t_1 t_2 \partial_{12} F + \frac{t_2^2}{2} \partial_{22} F + \dots$$

all the derivatives being taken at the point $t_1 = t_2 = 0$.

This will give

$$\mathcal{P}_n(\lambda) \sim e^{n\partial_2 F_0} e^{(N-n)\partial_{12} F_0} e^{-\frac{1}{2}\partial_{22} F_0} \frac{\theta_3(nx + (N-n)\partial_1 x - \partial_2 x)}{\theta_3(nx + (N-n)\partial_1 x)} (1 + O(N^{-1})) \quad (4.16)$$

Now, let us compute the derivatives of F_0 and $x = x_c$ with respect to t_1 and t_2 . The method proceeds similarly to section 3.1.

4.2.1 Derivatives of F_0 with respect to t_1 and t_2

using (4.15) with (4.13):

$$\frac{\partial F_0}{\partial t_2} = \frac{1}{2i\pi} \oint \omega(z) \ln(z - \lambda) dz$$

After integration by part, the pole in $(z - \lambda)$ picks a residue, and the result is a primitive of $\omega(\lambda)$:

$$\frac{\partial F_0}{\partial t_2} = \int_{\lambda_0}^{\lambda} \omega(z) dz \quad (4.17)$$

The lower bound of integration λ_0 is to be chosen such that $e^{n\partial_2 F_0} \sim_{\lambda \rightarrow \infty} \lambda^n$. i.e.

$$\ln \lambda_0 = \int_{\lambda_0}^{\infty} \left(\omega(z) - \frac{1}{z} \right) dz \quad (4.18)$$

In order to compute the second derivatives $\partial_{12} F_0$ and $\partial_{22} F_0$, we will need to differentiate $\omega(z)$ with respect to t_1 and t_2 .

4.2.2 Derivatives of $\omega(z)$ with respect to t_1 and t_2

The resolvent $\omega(z)$ computed for the potential $\mathcal{V}(z)$ takes the form:

$$\omega(z) = \frac{1}{2} \left(\mathcal{V}'(z) - M(z) \sqrt{\sigma(z)} \right) \quad (4.19)$$

where $M(z)$ is analytic. Notice that when $\mathcal{V}'(z)$ has a pole in $z = \lambda$, $M(z)$ may have a pole too. $\omega(z)$ obeys a linear equation:

$$\omega(z + i0) + \omega(z - i0) = \mathcal{V}'(z) \quad \text{for} \quad z \in [a, b] \cup [c, d] \quad (4.20)$$

Thus its derivatives obey linear equations as well:

$$\partial_1 \omega(z + i0) + \partial_1 \omega(z - i0) = \delta V_1'(z) = V'(z) \quad (4.21)$$

$$\partial_2\omega(z+i0) + \partial_2\omega(z-i0) = \delta V_2'(z) = \frac{1}{z-\lambda} \quad (4.22)$$

• $\partial\omega_1$: The solution of (4.21) is:

$$\partial_1\omega(z) = \omega(z) - \frac{f(z)}{\sqrt{\sigma(z)}} \quad (4.23)$$

where $f(z)$ is analytic in z . The boundary conditions 2.18 imply that $f(z) \sim z$ when $z \rightarrow \infty$ and f has no pole, thus $f(z)$ is a polynomial of degree 1:

$$\frac{\partial\omega(z)}{\partial t_1} = \omega(z) - \frac{z-z_0}{\sqrt{\sigma(z)}} \quad (4.24)$$

z_0 is determined as a function of a, b, c, d by the derivative of 2.30 with respect to t_1 :

$$\int_b^c dz \frac{z-z_0}{\sqrt{\sigma(z)}} = 0 \quad (4.25)$$

It can be checked that in term of elliptic theta functions we have (see Appendix A, or [23]):

$$\frac{z-z_0}{\sqrt{\sigma(z)}} = \frac{d}{dz} \ln \frac{\theta_1(u(z)+u_\infty)}{\theta_1(u(z)-u_\infty)} \quad (4.26)$$

and thus:

$$\frac{\partial\omega(z)}{\partial t_1} = \omega(z) - \frac{d}{dz} \ln \frac{\theta_1(u(z)+u_\infty)}{\theta_1(u(z)-u_\infty)} \quad (4.27)$$

• $\partial\omega_2$: Note that the t_2 source-term is the primitive of the ϵ_λ source-term of 3.2, and that

$$\frac{d}{dz} \frac{\partial\omega(z)}{\partial t_2} = \frac{\partial\omega(z)}{\partial \epsilon_\lambda} = -\frac{1}{n^2} \frac{\partial^2 F}{\partial \epsilon_z \partial \epsilon_\lambda} = \omega_c(z, \lambda) \quad (4.28)$$

so, $\partial\omega/\partial t_2$ has already been computed in 3.8. The second derivative $\partial_{22}F$ corresponds to $z = \lambda$.

$$\partial_{22}F_0 = \ln \sqrt{\sigma(\lambda)} + 2 \ln (\theta_1(u(\lambda) - u_\infty)) \quad (4.29)$$

4.2.3 Derivatives of x

Recall that

$$x = \int_a^b \rho(\lambda) d\lambda = \frac{1}{2i\pi} \int_a^b M(z) \sqrt{\sigma(z)} dz \quad (4.30)$$

using 4.27 we get:

$$\frac{\partial x}{\partial t_1} = x + \frac{1}{2i\pi} \int_a^b \frac{z-z_0}{\sqrt{\sigma(z)}} dz = x + 2u_\infty \quad (4.31)$$

Similarly, from 4.28 and 3.18 or 3.31 (or less tediously, taking the primitive of 3.14), we get:

$$\frac{\partial x}{\partial t_2} = -u(\lambda) + u_\infty \quad (4.32)$$

4.3 Final result

4.3.1 Case $\lambda \notin [a, b] \cup [c, d]$

Eventually, inserting 4.17,4.27,4.29,4.31, 4.32 into 4.16 we get:

$$\mathcal{P}_n(\lambda) \underset{\lambda \notin [a,b] \cup [c,d]}{=} \sqrt{u'(\lambda)} p_n(u(\lambda)) e^{N \int_{\lambda_0}^{\lambda} \omega} \quad (4.33)$$

where

$$p_n(u) = C_n \frac{\theta_3(Nx + 2(N-n)u_\infty + u - u_\infty)\theta_1(2u_\infty)}{\theta_3(Nx + 2(N-n)u_\infty)\theta_1(u - u_\infty)} \left(\frac{\theta_1(u + u_\infty)}{\theta_1(u - u_\infty)} \right)^{n-N} \quad (4.34)$$

C_n is a normalization such that $\mathcal{P}_n \sim \lambda^n$ for $\lambda \rightarrow \infty$.

$$C_n = \sqrt{2K} A^{n-N+1} \quad \text{with} \quad A = -\frac{1}{2K} \frac{\theta_1'(0)}{\theta_1(2u_\infty)} \quad \text{and} \quad K = \int_c^b \frac{dz}{\sqrt{\sigma(z)}} \quad (4.35)$$

$$A = -\frac{1}{4} |d - a - c + b| \frac{\theta_3(0)}{\theta_3(2u_\infty)}$$

Note that 4.33 is unchanged under $u \rightarrow u + 1$ and $u \rightarrow u + \tau$. Indeed, a shift $u \rightarrow u + \tau$ amounts to a nontrivial circle around the cut $[c, d]$. Thus $\int \omega$ is shifted by $-2i\pi \int_c^d \rho = -2i\pi(1-x)$, and $e^{N \int \omega}$ receives a phase $e^{2i\pi Nx}$. In the same time, the θ functions receive phase factors: $\theta(v + \tau) = \theta(v)e^{-2i\pi(v+\tau/2)}$. One can easily check that the total phase shift is 0.

4.3.2 Case $\lambda \in [a, b] \cup [c, d]$

Expression 4.34 has been derived by a saddle point approximation of 4.12 when λ does not belong to $[a, b] \cup [c, d]$. When λ lies on the cut $[a, b] \cup [c, d]$, 4.12 actually has two saddle points, contributing to the same order. They correspond to the two determinations of the square root $\pm\sqrt{\sigma(\lambda)}$. The asymptotic expression for the orthogonal polynomial is then given by a sum of two terms:

$$\mathcal{P}_n(\lambda) \underset{\lambda \in [a,b] \cup [c,d]}{=} C \sqrt{u'} [p_n(u) e^{-iN\pi\zeta(\lambda)} + ip_n(-u) e^{iN\pi\zeta(\lambda)}] e^{\frac{N}{2}V(\lambda)} \quad (4.36)$$

where $\zeta(\lambda) = \int_d^\lambda \rho(z)dz$ and

$$C = e^{-\frac{N}{2}(V(\lambda_0) + \int_{\lambda_0}^d M(z)\sqrt{\sigma(z)}dz)} = d^N e^{-\frac{N}{2}V(d)} e^{-N \int_d^\infty (\omega(z) - \frac{1}{z})dz} \quad (4.37)$$

To summarize: When $\lambda \notin [a, b] \cup [c, d]$, the wave function $\psi_n(\lambda) = \mathcal{P}_n(\lambda)e^{-NV/2}$ decays exponentially, and within the support $[a, b] \cup [c, d]$, it oscillates at a frequency of order N .

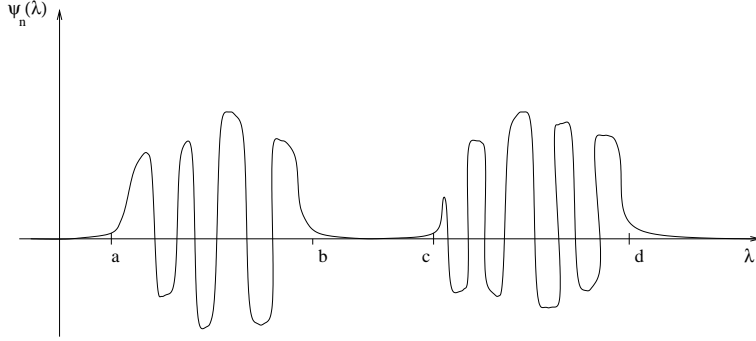


Figure 2: Typical behavior of the wave function

4.3.3 Check of the orthogonality

For completeness, let us check that the functions (4.33) are indeed orthogonal (at leading order in N^{-1}). Let us compute the integral:

$$\int_{-\infty}^{\infty} d\lambda \mathcal{P}_n(\lambda) \mathcal{P}_m(\lambda) e^{-NV(\lambda)}$$

The contributions of the integral along $]-\infty, a[\cup [b, c] \cup [d, \infty[$ are exponentially small and do not contribute at leading order.

Along $[a, b] \cup [c, d]$, we use expression 4.36, and get a sum of four terms:

$$C^2 \int d\lambda u'(\lambda) \begin{pmatrix} ip_n(u)p_m(-u) & +ip_n(-u)p_m(u) \\ +p_n(u)p_m(u)e^{-2iN\pi\zeta(\lambda)} & -p_n(-u)p_m(-u)e^{2iN\pi\zeta(\lambda)} \end{pmatrix} \quad (4.38)$$

Since the two last terms have fast oscillations of frequency N , they are suppressed as $O(1/N)$.

The leading contribution is thus given by the two first terms of 4.38, which can be rewritten as integrals in the u plane along the contour depicted on fig.3.a:

$$\int \mathcal{P}_n \mathcal{P}_m e^{-NV} = ic_{nm} \int du \frac{\theta_3(x_n + u - u_\infty) \theta_3(x_m - u - u_\infty)}{\theta_3(x_n) \theta_3(x_m) \theta_1(u - u_\infty) \theta_1(u + u_\infty)} \left(\frac{\theta_1(u + u_\infty)}{\theta_1(u - u_\infty)} \right)^{n-m} + (u \rightarrow -u) \quad (4.39)$$

where x_n and c_{nm} are short notations for:

$$x_n = Nx + 2(N - n)u_\infty \quad \text{and} \quad c_{nm} = C^2 C_n C_m \theta_1^2(2u_\infty) \quad (4.40)$$

If $n > m$ we may deform the contour to a circle around the point $-u_\infty$ (fig.3.b), and the integral vanishes since there is no pole, while if $m > n$ we deform the contour to a circle around $+u_\infty$ (fig.3.c). Therefore, the integral vanishes for $n \neq m$.

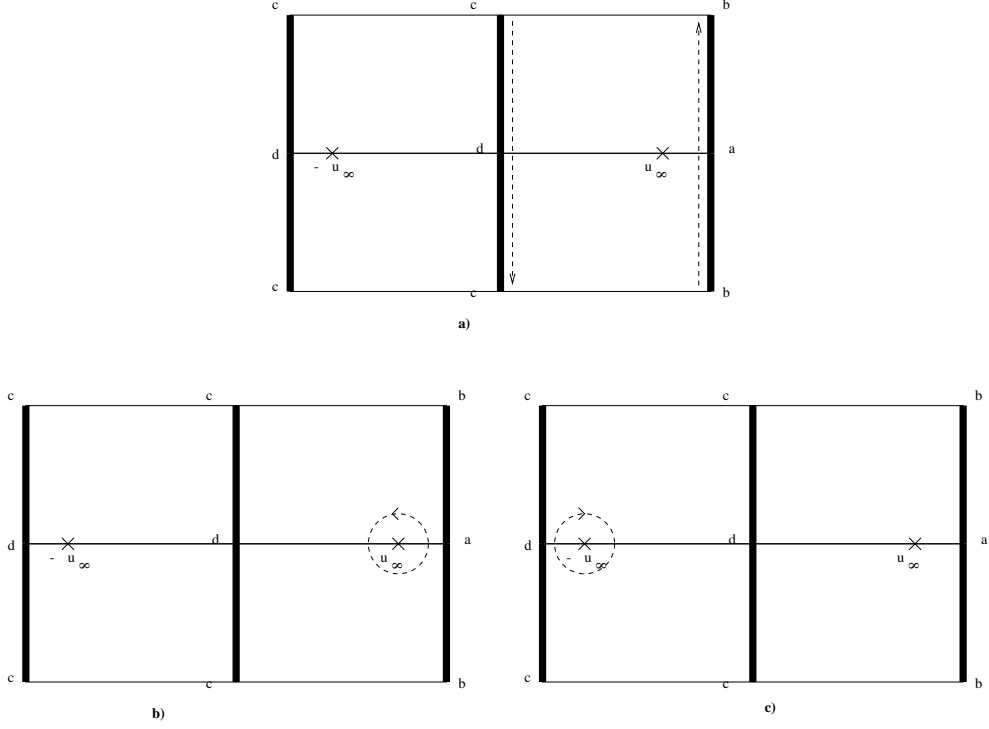


Figure 3: Deformation of the contour integral

When $n = m$, the integral picks a residue:

$$\int d\lambda \mathcal{P}_n \mathcal{P}_m e^{-NV(\lambda)} = h_n \delta_{nm} \quad (4.41)$$

with $(C, c_{nm}, A, x_n$ are defined in 4.35, 4.40, 4.37):

$$h_n = c_{nn} \frac{4\pi}{\theta_1(2u_\infty)\theta_1'(0)} \frac{\theta_3(x_{n+1})}{\theta_3(x_n)} = -4\pi C^2 \frac{\theta_3(x_{n+1})}{\theta_3(x_n)} A^{2(n-N+1/2)} \quad (4.42)$$

4.3.4 Recurrence equation

It is well known that the orthogonal polynomials satisfy a recurrence equation of the form [2, 18]:

$$\lambda \mathcal{P}_n(\lambda) = \mathcal{P}_{n+1}(\lambda) + \beta_n \mathcal{P}_n(\lambda) + \alpha_n \mathcal{P}_{n-1}(\lambda) \quad (4.43)$$

Here, we find that (divide 4.43 by \mathcal{P}_n , and match the poles on both sides):

$$\alpha_n = \frac{h_n}{h_{n-1}} = A^2 \frac{\theta_3(x_{n+1})\theta_3(x_{n-1})}{\theta_3^2(x_n)} \quad (4.44)$$

which can be rewritten more compactly as

$$\alpha_n = \frac{1}{16} \left(((d-a) - (c-b))^2 + 4(d-a)(c-b) \operatorname{cn}^2(x_n + 1/2) \right) \quad (4.45)$$

And by taking $u = 0$ in 4.43 we get β_n :

$$\beta_n - d = A \left[\frac{\theta_3(x_{n+3/2})\theta_3(x_n)}{\theta_3(x_{n+1})\theta_3(x_{n+1/2})} + \frac{\theta_3(x_{n+1})\theta_3(x_{n-1/2})}{\theta_3(x_n)\theta_3(x_{n+1/2})} \right] \quad (4.46)$$

which can be rewritten more compactly as:

$$\beta_n = \frac{a + d + (c - b)}{2} - (c - b) \frac{d - b}{c - b + \frac{d - c}{\text{cn}^2(x_n - u_\infty + 1/2)}} \quad (4.47)$$

The sequences α_n and β_n are thus quasi-periodic in n . It is interesting to recall that the behavior of these coefficients has been extensively studied (mainly by numerical methods) by several authors in the early 90's [22]. The general conclusion was that in the multi-cut case the general behavior of the recursion coefficients was ‘‘chaotic’’ in n (and regular or quasi-periodic only in some special cases). It is clear from our expressions that in the two-cut case the behavior is **always** periodic or quasi-periodic and never chaotic (in the mathematical sense). This is in fact true even if the number of cuts is larger than 2 (see appendix C).

In the symmetric case, $x = \frac{1}{2}$ and $u_\infty = \frac{1}{4}$, we have $x_n = n/2 \bmod 1$, so that we recover $\beta_n = 0$ and $\alpha_n = \frac{1}{4}(a - (-1)^n b)^2$.

In the general case, α_n and β_n vary along a periodic curve, between two extrema, given by:

$$\begin{aligned} \frac{(d - a - (c - b))^2}{16} &\leq \alpha_n \leq \frac{(d - a + (c - b))^2}{16} \\ \frac{d + a}{2} - \frac{c - b}{2} &\leq \beta_n \leq \frac{d + a}{2} + \frac{c - b}{2} \end{aligned}$$

Similarly to the one-cut case, one may relate α_n to square width of the distribution of eigenvalues, and β_n to the center of the distribution.

4.4 The kernel $K(\lambda, \mu)$

We can now evaluate the kernel $K(\lambda, \mu)$ according to (4.10). Let us note $u = u(\lambda)$ and $v = u(\mu)$ and we assume $\lambda, \mu \in [a, b] \cup [c, d]$:

$$K(\lambda, \mu) \sim \frac{C^2 \sqrt{u'v'}}{N h_{N-1}} \sum_{\epsilon, \eta = \pm 1} \frac{\sqrt{\epsilon \eta} p_N(\epsilon u) p_{N-1}(\eta v) e^{-\epsilon N i \pi \zeta(\lambda)} e^{-\eta N i \pi \zeta(\mu)} - (u \rightarrow v)}{(\lambda - \mu)} \quad (4.48)$$

which can be rewritten as a sum of eight terms:

$$K(\lambda, \mu) = \sum_{\epsilon, \eta, \kappa = \pm 1} \kappa \sqrt{\epsilon \eta} \frac{c_{N, N-1}}{h_{N-1} \theta_3(x_N) \theta_3(x_{N-1})} \frac{\sqrt{u'v'}}{N(\lambda - \mu)} \times \frac{\theta_3(Nx + \epsilon u - \kappa u_\infty) \theta_3(Nx + \eta v + \kappa u_\infty)}{\theta_1(\epsilon u - \kappa u_\infty) \theta_1(\eta v + \kappa u_\infty)} e^{-N i \pi (\epsilon \zeta(\lambda) + \eta \zeta(\mu))} \quad (4.49)$$

We will see below that not all the terms contribute to the same order.

4.4.1 Regime $|\lambda - \mu| \sim O(1/N)$

The eight terms of (4.49) can be rewritten as four combinations of the type:

$$\sin(N\pi(\zeta(\lambda) \pm \zeta(\mu))) \frac{f(u, v) \mp f(v, u)}{N(\lambda - \mu)} \quad \text{and} \quad \cos(N\pi(\zeta(\lambda) \pm \zeta(\mu))) \frac{g(u, v) - g(v, u)}{N(\lambda - \mu)}$$

In the limit $|\lambda - \mu|$ small, i.e. $|u - v|$ small, the terms with a cosine will be proportional to derivatives of $g(u, v)$, and there will be an overall $\frac{1}{N}$ factor. Similarly, the term with a sine and a + sign will be proportional to a derivative of $f(u, v)$ and will be of order $1/N$. Only the term proportional to $\sin N\pi \int_{\lambda}^{\mu} \rho(z) dz$ can balance the $1/N$ factor, and is dominant in the short range regime. After calculation we get:

$$K(\lambda, \mu) \Big|_{|\lambda - \mu| \sim O(1/N)} \sim \frac{\sin N\pi \int_{\lambda}^{\mu} \rho(z) dz}{N\pi(\lambda - \mu)} \quad (4.50)$$

As expected we have

$$K(\lambda, \lambda) = \rho(\lambda) \quad (4.51)$$

and we recover the universal short range correlation function:

$$\rho(\lambda, \mu) \sim \rho(\lambda)\rho(\mu) \left(1 - \left(\frac{\sin N\pi\rho(\lambda)(\lambda - \mu)}{N\pi\rho(\lambda)(\lambda - \mu)} \right)^2 \right) \quad (4.52)$$

4.4.2 Long range regime, smoothed oscillations

When $|\lambda - \mu| \sim O(1)$, $K(\lambda, \mu)$ has high frequency oscillations, and only a smoothed correlation function obtained by averaging the oscillations can be observed.

Recall that the connected 2-point correlation function is related to K^2 by (4.9):

$$\rho_{2c}(\lambda, \mu) = -K(\lambda, \mu)^2 \quad (4.53)$$

with $K(\lambda, \mu)$ given by 4.49.

Smoothing out the oscillations amounts to kill all terms containing some $e^{iN\pi\zeta}$ in the square of eq 4.49, we thus have:

$$\overline{K(\lambda, \mu)^2} = \frac{-2u'v'c_{NN-1}^2}{h_{N-1}^2\theta_3^2(x_N)\theta_3^2(x_{N-1})N^2(\lambda - \mu)^2} \sum_{\epsilon, \eta, \kappa_1, \kappa_2 = \pm 1} \kappa_1\kappa_2 \frac{\theta_3(Nx + \epsilon u - \kappa_1 u_\infty)\theta_3(Nx - \epsilon u - \kappa_2 u_\infty)\theta_3(Nx + \eta v + \kappa_1 u_\infty)\theta_3(Nx - \eta v + \kappa_2 u_\infty)}{\theta_1(\epsilon u - \kappa_1 u_\infty)\theta_1(-\epsilon u - \kappa_2 u_\infty)\theta_1(\eta v + \kappa_1 u_\infty)\theta_1(-\eta v + \kappa_2 u_\infty)} \quad (4.54)$$

Using that (see appendix A, and [23, 25, 24]):

$$\lambda - \mu = -2A \frac{\theta_1(u-v)\theta_1(u+v)\theta_1^2(2u_\infty)}{\theta_1(u-u_\infty)\theta_1(u+u_\infty)\theta_1(v-u_\infty)\theta_1(v+u_\infty)} \quad (4.55)$$

we get ($\epsilon_{ij} = -\epsilon_{ji} = \pm 1$):

$$\begin{aligned} \overline{K(\lambda, \mu)^2} &= \frac{2u'v'\theta_1^2(0)}{N^2 4\pi^2 \theta_3^4(Nx)\theta_1^2(2u_\infty)} \\ &\left(\frac{\theta_1(u-u_\infty)\theta_1(u+u_\infty)\theta_1(v-u_\infty)\theta_1(v+u_\infty)}{\theta_1(u-v)\theta_1(u+v)} \right)^2 \sum_{i,j,k,l=\pm 1} (\epsilon_{ij}\epsilon_{kl} + \epsilon_{il}\epsilon_{kj}) \quad (4.56) \\ &\frac{\theta_3(Nx+u-iu_\infty)\theta_3(Nx-u-ku_\infty)\theta_3(Nx+v-ju_\infty)\theta_3(Nx-v-lu_\infty)}{\theta_1(u-iu_\infty)\theta_1(-u-ku_\infty)\theta_1(v-ju_\infty)\theta_1(-v-lu_\infty)} \end{aligned}$$

We see that 4.56 has no pole when $u = \pm u_\infty$, it can have (double) poles only when $u = \pm v$. Thus, 4.56 can be rewritten in terms of Weirstrass functions of $u+v$ and $u-v$:

$$\overline{K(\lambda, \mu)^2} = -\frac{1}{2N^2\pi^2} u'v' (C_1\wp(u-v) + C_2\wp(u+v) - 2S)$$

Taking $u = v$ and $u = -v$ in 4.56, we find that the residues are $C_1 = C_2 = 1$, and taking a particular value of u and v , we find the constant S , equal to what we had in 3.31:

$$\overline{K(\lambda, \mu)^2} = -\frac{1}{2N^2\pi^2} u'v' \left(\wp(u-v) + \wp(u+v) - 2\wp(Nx + \frac{\tau}{2}) \right) \quad (4.57)$$

and we recover the result 3.31 found in section 3.6.

5 Conclusions

In this article, we have solved the puzzle raised by [11, 13] and understood why the naive mean-field method [10] and the orthogonal polynomial ansatz [11, 12] approach used in the symmetric case disagree.

We have proven here that this effect has nothing to do with a \mathbb{Z}_2 symmetry breaking, as it was sometimes assumed [11], it is general as soon as the support of the density is not connected.

The apparent paradox comes from the fact that when the support of eigenvalues is not connected, the free energy admits no large N expansion in powers of $1/N^2$ (topological expansion [9]). This means that the free energy in the multi-cut case is not given by a topological expansion, i.e. the sum of diagrams with a weight N^χ (χ =Euler Characteristic of the diagram).

The explanation lies in the discreteness of the number of eigenvalues. For instance in the symmetric 2-cut case, the classical approach assumes that the minimum of the free energy is reached when one half ($x = 1/2$) of the eigenvalues are in each cut. Obviously, this minimum is never reached when the total number of eigenvalues is odd, and in general, the result depends on the fractional part of Nx .

At leading order in N only, the free energy is correctly given by the classical saddle point limit [16, 10], but the first order in N is not sufficient to determine the 2-point (or higher) correlation function.

Here we have computed explicitly the two-point connected correlation function. It contains a universal part depending only on the number of cuts, which was obtained by [10], and contains in addition, a non universal term quasiperiodic in N [22, 14].

Let us stress that our calculation holds for any potential, not necessarily symmetric, and it can also be generalized to a potential with complex coefficients (appendix. B), and to an arbitrary number of cuts (appendix. C).

We have also reobtained directly the asymptotic expressions for the orthogonal polynomials [14], which allows in principle through the Darboux-Christoffel theorem (eq. 4.10) to compute any correlation function of any number of eigenvalues in the short or long range domain (and one can smooth it afterwards).

The orthogonal polynomial approach may in turn be used for other random matrix ensembles, and it would be interesting to apply our results to orthogonal or symplectic ensembles [2].

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Appendix A A few useful identities on elliptic functions

Here we collect a few useful identities on elliptic functions used through the paper. For details see [23, 24, 25]. We start from

$$\sigma(\lambda) = (\lambda - a)(\lambda - b)(\lambda - c)(\lambda - d) \quad , \quad a < b < c < d \quad (\text{A.1})$$

and the map from the complex plane to the torus

$$u(\lambda) = \frac{1}{2K} \int_d^\lambda \frac{dz}{\sqrt{\sigma(z)}} \quad (\text{A.2})$$

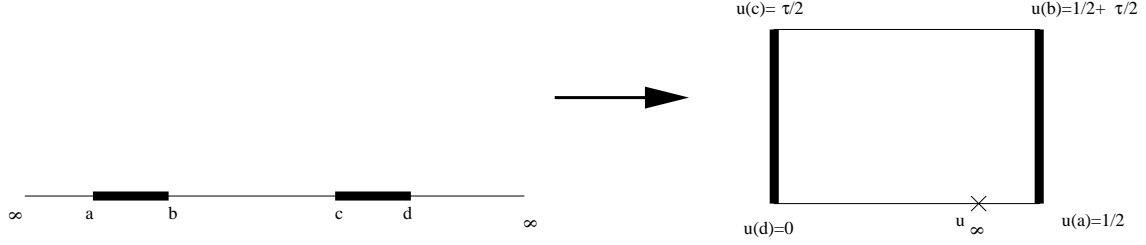


Figure 4: the upper half-plane is mapped onto a rectangle $(1/2, \tau/2)$

where the half period K is

$$K = \int_b^c \frac{dz}{\sqrt{|\sigma(z)|}} = \frac{2}{\sqrt{(c-a)(d-b)}} K[m] = \frac{\pi \theta_3^2(0|\tau)}{\sqrt{(c-a)(d-b)}} \quad (\text{A.3})$$

$K[m]$ is the standard complete elliptic integral [23, 24, 25], with the modulus m equal to the biratio of the four points a, b, c, d :

$$m = \frac{(d-a)(c-b)}{(d-b)(c-a)} \quad (\text{A.4})$$

m is related to the modular parameter τ of the torus by:

$$m = e^{i\pi\tau} \frac{\theta_3^4(\frac{\tau}{2}|\tau)}{\theta_3^4(0|\tau)} \quad \text{and conversely} \quad \tau = i \frac{K[1-m]}{K[m]} \quad (\text{A.5})$$

where we have used the Jacobi theta functions:

$$\theta_1(z|\tau) = \theta_1(z) = -i \sum_{r \in \mathbb{Z} + 1/2} (-1)^r q^{r^2} e^{2i\pi r z} \quad \text{with} \quad q = e^{i\pi\tau} \quad (\text{A.6})$$

$$\text{and} \quad \theta_3(z|\tau) = \theta_3(z) = q^{\frac{1}{4}} e^{i\pi z} \theta_1(z + \frac{1}{2} + \frac{\tau}{2}|\tau) \quad (\text{A.7})$$

With this mapping $u(\lambda)$ between the λ complex plane and the periodic rectangle of sides $(1, \tau)$, we have:

$$u(d) = 0 \quad , \quad u(a) = \frac{1}{2} \quad , \quad u(b) = \frac{1+\tau}{2} \quad , \quad u(c) = \frac{\tau}{2} \quad , \quad u(\infty) = u_\infty \quad (\text{A.8})$$

The inverse mapping can be written in terms of theta functions:

$$\lambda - d = -\frac{\theta_1'(0)}{2K} \frac{\theta_1^2(u)\theta_1(2u_\infty)}{\theta_1(u+u_\infty)\theta_1(u-u_\infty)\theta_1^2(u_\infty)} \quad (\text{A.9})$$

$$\sqrt{\sigma(\lambda)} = \frac{\theta_1^2(0)}{4K^2} \frac{\theta_1(2u)\theta_1(2u_\infty)}{\theta_1^2(u-u_\infty)\theta_1^2(u+u_\infty)} \quad (\text{A.10})$$

and in terms of the usual trigonometric elliptic functions sn , cn , dn [23, 24, 25] that we normalize to have periods 1 and τ , i.e.

$$\text{sn}(u) = \text{sn}(2K[m]u|m) \quad , \quad \text{dn}(u) = \text{dn}(2K[m]u|m) \quad , \quad \dots \quad (\text{A.11})$$

one has

$$\frac{\text{sn}^2(u)}{\text{sn}^2(u_\infty)} = \frac{\lambda - d}{\lambda - c} \quad , \quad \frac{\text{cn}^2(u)}{\text{cn}^2(u_\infty)} = \frac{\lambda - a}{\lambda - c} \quad , \quad \frac{\text{dn}^2(u)}{\text{dn}^2(u_\infty)} = \frac{\lambda - b}{\lambda - c} \quad (\text{A.12})$$

$$\lambda - c = (d - c) \frac{1}{1 - \frac{\text{sn}^2(u)}{\text{sn}^2(u_\infty)}} \quad , \quad \lambda - b = (d - b) \frac{\text{dn}^2(u)}{1 - \frac{\text{sn}^2(u)}{\text{sn}^2(u_\infty)}} \quad (\text{A.13})$$

$$\lambda - a = (d - a) \frac{\text{cn}^2(u)}{1 - \frac{\text{sn}^2(u)}{\text{sn}^2(u_\infty)}} \quad , \quad \lambda - d = (d - c) \frac{d - a}{c - a} \frac{\text{sn}^2(u)}{1 - \frac{\text{sn}^2(u)}{\text{sn}^2(u_\infty)}} \quad (\text{A.14})$$

$$\sqrt{\sigma(\lambda)} = (d - c)(d - a) \sqrt{\frac{d - b}{c - a} \frac{\text{sn}(u) \text{cn}(u) \text{dn}(u)}{\left(1 - \frac{\text{sn}^2(u)}{\text{sn}^2(u_\infty)}\right)^2}} \quad (\text{A.15})$$

and u_∞ is related to a, b, c, d by any of the following relations:

$$\text{sn}^2(u_\infty) = \frac{c - a}{d - a} \quad , \quad \text{cn}^2(u_\infty) = \frac{d - c}{d - a} \quad , \quad \text{dn}^2(u_\infty) = \frac{d - c}{d - b} \quad (\text{A.16})$$

Appendix B Complex potentials

The case of complex potentials, that is to say of a polynomial potential $V(\lambda)$ with complex coefficients, is interesting for some applications of the matrix models to 2 dimensional gravity and when studying their connections with integrable hierarchies. In this case, the mean field large N solution is known to be given by a continuous distribution of the eigenvalues along arcs in the complex plane [26].

In this appendix we show that our results are only slightly modified in this case.

In the two-cut case, we can repeat the analysis of sect.2. We fix $x = n_1/N$ (the proportion of e.v. in the first cut). The resolvent is still of the form 2.28, with the polynomial M and the end points a, b, c, d determined by the constraints 2.18 and 2.29, but they are no more real in general, as well as the resulting mean-field free energy $F_0(x)$.

If we now repeat the calculation of sect.2.2.3 we cannot use a saddle-point approximation for the sum over n_1 by expanding $F_0(x)$ around the saddle point x_0 which is the true extremum of F_0 .

$$\frac{\partial F_0}{\partial x}(x_0) = 0 \quad (\text{B.1})$$

Indeed this extremum is at a finite non-zero distance of the real axis, i.e. $\text{Im}(x_0) = \mathcal{O}(1)$, while the method of sect.2.2.3 is valid only if $\text{Im}(x_0) = \mathcal{O}(1/N)$. However, since N is integer, we can expand F_0 around any x_k , provided that

$$F'_0(x_k) = 2i\pi \frac{k}{N} \quad , \quad k \in \mathbb{Z} \quad (\text{B.2})$$

since the dangerous oscillating term $e^{-N^2(x-x_k)F'_0(x_c)}$ is then a constant for $x = n/N$, $n \in \mathbb{Z}$.

Therefore, as in [26] we have to consider the *real pseudo saddle-point* x_c such that

$$\text{Re}(F'_0(x_c)) = 0 \quad \text{with} \quad \text{Im}(x_c) = 0 \quad (\text{B.3})$$

and denote

$$\Delta_c = \frac{1}{2i\pi} F'_0(x_c) = \frac{1}{2\pi} \text{Im}(F'_0(x_c)) \quad (\text{B.4})$$

We expand F_0 around some x_k defined by Eq. (B.2) and such that

$$x_k - x_c = \mathcal{O}(N^{-1}) \quad (\text{B.5})$$

and we get for the total free energy (by exactly the same calculation as in sect.2.2.3)

$$\begin{aligned} F &= N^2 F_0(x_k) - \ln(\theta_3(Nx_k)) \\ &+ F_1(x_c) + \frac{1}{2} \ln(2\pi F''_0(x_c)) + \mathcal{O}(N^{-2}) \end{aligned} \quad (\text{B.6})$$

where θ_3 is the theta function with modular parameter

$$\tau = \frac{2i\pi}{F''_0(x_c)} \quad (\text{B.7})$$

Only the first two terms are important for calculating the two-point functions and the orthogonal polynomials in the large N limit. This leading term does not depend on k . Indeed we have

$$F'_0(x_k) - F'_0(x_c) = (x_k - x_c)F''_0(x_c) + \mathcal{O}(N^{-1}) \quad (\text{B.8})$$

hence

$$(x_k - x_c) = \frac{1}{N} \tau(k - N\Delta_c) + \mathcal{O}(N^{-2}) \quad (\text{B.9})$$

and using the periodicity relations of θ_3 we can rewrite the leading term for the free energy as

$$N^2 F_0(x_k) - \ln(\theta_3(Nx_k)) = N^2 F_0(n_c/N) - \frac{i\pi}{\tau} u_c^2 - \ln(\theta_3(u_c)) \quad (\text{B.10})$$

with

$$n_c = \text{E}[Nx_c] \quad , \quad u_c = [Nx_c] - \tau [N\Delta_c] \quad (\text{B.11})$$

where $E[u]$ is the integer part of u (largest integer smaller than u) and $[u] = u - E[u]$ is the fractional part of u . This does not depend on k up to negligible terms of order $\mathcal{O}(N^{-1})$ (provided that condition B.5 for k holds).

One can now repeat the calculation of sect.3 for the 2-point function. Nothing is changed but we simply have to replace x_c by x_k in the intermediate steps and to use Eq. (B.9) at the end of the calculation. This amounts to replace the x_c in the elliptic function sn^2 by $x_c - \tau\Delta_c$. The final result for the two-point resolvent is

$$\begin{aligned} \omega^c(\lambda, \mu) = & -\frac{1}{4(\lambda - \mu)^2} \left[\left(1 - \sqrt{\frac{(\lambda - a)(\lambda - b)(\mu - c)(\mu - d)}{(\mu - a)(\mu - b)(\lambda - c)(\lambda - d)}} \right) + (\lambda \leftrightarrow \mu) \right] \\ & - \frac{(c - a)(d - b)}{4\sqrt{\sigma(\lambda)}\sqrt{\sigma(\mu)}} \text{sn}^2(N(x_c - \tau\Delta_c) + \tfrac{1}{2}) \end{aligned} \quad (\text{B.12})$$

Similar results holds for the orthogonal polynomials. We simply have to consider the end-points a, b, c, d for the mean-field real parameter x_c and to make the replacement

$$Nx_c \rightarrow N(x_c + \tau\Delta_c) \quad (\text{B.13})$$

in the elliptic functions involving Nx_c . In any case these terms depend only on the fractional parts of Nx_c and of $N\Delta_c$.

A final interesting remark on the periodicity properties of the non-universal term $\text{sn}^2(N(x_c + \tau\Delta_c))$ can be made. From the definition B.4 Δ_c corresponds to a “phase shift” between the two arcs where the density of e.v. is non zero.

$$\Delta_c = \frac{1}{2i\pi} F'_0(x_c) = \frac{\Gamma_1 - \Gamma_2}{2i\pi} \quad (\text{B.14})$$

where $N\Gamma_\alpha$ is the (constant) effective potential on the arc α . The two periods 1 and τ of the sn^2 function correspond respectively in term of eigenvalues to (i) transfer a single e.v. from the first arc to the second one ($\delta Nx = \pm 1$), (ii) or to shift the phase between the two arcs by 2π ($\delta N\Delta_c = \pm 1$).

Appendix C Multicut Case

Consider now a support of eigenvalues split into s intervals:

$$\mathcal{C} = \mathcal{C}_1 \cup \dots \cup \mathcal{C}_s \quad (\text{C.1})$$

Let n_i be the number of eigenvalues in each \mathcal{C}_i , and $x_i = n_i/N$ the occupation ratio, which we denote collectively as a vector:

$$x_i = \int_{\mathcal{C}_i} d\lambda \rho(\lambda) \quad , \quad \vec{x} = (x_1, \dots, x_{s-1}) \quad (\text{C.2})$$

Note that only $s - 1$ of them are independent since $x_1 + \dots + x_s = 1$.

As in the two-cut case (eq. 2.36), the free energy at fixed \vec{n} admits a topological large N expansion:

$$F[V; \vec{n}] = N^2 F_0[V; \vec{x}] + N^0 F_1[V; \vec{x}] + \mathcal{O}(1/N^2) \quad (\text{C.3})$$

and as in 2.38, the partition function can be written as a sum over \vec{n} :

$$Z = e^{-F} = \sum_{\vec{n}} e^{-F[V, \vec{n}]} \quad (\text{C.4})$$

The sum is dominated by the vicinity of the extremum \vec{x}_c of $F_0[V; \vec{x}]$:

$$Z \sim \sum_{\vec{n}} e^{-N^2 (F_0[V, \vec{x}_c] + i\pi (\frac{\vec{n}}{N} - \vec{x}_c) \cdot \tau^{-1} (\frac{\vec{n}}{N} - \vec{x}_c))} \quad \text{where} \quad \left. \frac{\partial}{\partial \vec{x}} F_0(\vec{x}) \right|_{\vec{x}=\vec{x}_c} = \vec{0} \quad (\text{C.5})$$

τ is the $s - 1 \times s - 1$ matrix defined by:

$$\tau_{ij}^{-1} = \frac{1}{2i\pi} \left. \frac{\partial^2 F_0}{\partial x_i \partial x_j} \right|_{\vec{x}=\vec{x}_c} \quad (\text{C.6})$$

Then, the summation over \vec{n} yields:

$$Z \sim e^{-N^2 F_0[V, \vec{x}_c]} \theta(N \vec{x}_c | \tau) \quad (\text{C.7})$$

where $\theta(\vec{u} | \tau)$ is Riemann's theta function [27] in genus $s - 1$:

$$\theta(\vec{u} | \tau) = \theta(\vec{u}) = \sum_{\vec{n}} e^{-i\pi (\vec{n} - \vec{u}) \cdot \tau^{-1} (\vec{n} - \vec{u})} = \sum_{\vec{n}} e^{i\pi \vec{n} \cdot \tau \vec{n}} e^{-2i\pi \vec{n} \cdot \vec{u}} \quad (\text{C.8})$$

where τ is a $s - 1 \times s - 1$ matrix, \vec{u} is a $s - 1$ component vector, and \vec{n} is a vector with integer coordinates.

The θ function obeys the relations (\vec{k} being an arbitrary integer vector):

$$\theta(\vec{u} + \vec{k}) = \theta(\vec{u}) \quad , \quad \theta(\vec{u} + \tau \vec{k}) = e^{-i\pi (2\vec{u} \cdot \vec{k} + \vec{k} \cdot \tau \vec{k})} \theta(\vec{u}) \quad , \quad \theta(-\vec{u}) = \theta(\vec{u}) \quad (\text{C.9})$$

Eventually the free energy at leading orders in N is:

$$F \sim N^2 \left[F_0[V, \vec{x}_c] - \frac{1}{N^2} \ln \theta(N \vec{x}_c | \tau) + \frac{1}{N^2} F_1(\vec{x}_c) + \dots \right] \quad (\text{C.10})$$

It is now straightforward but lengthy to rederive the 2-point correlation function and the orthogonal polynomials from C.10. One needs to differentiate C.10 with respect to variations of the potential as in 3.3 or 4.16, and express the hyperelliptical functions involved in the calculation through prime forms (hyperelliptical generalization of the θ_1 function) [27]. One should thus obtain expressions similar to those of [14].

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