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## Appendix 1: Worked Examples

### 1. The value of ${}_1m_0 = 0.05$ —for both sexes combined—is given and a previous study concluded that $k = 0.5$ is best representing the age patterns of under-five mortality in this population

Step 1: predict a mortality schedule  $q(x; {}_5q_0, k)$  for a feasible combination of parameters, using the Equation 2 defined in the paper. As an initial guess, we have assumed an arbitrary level of under-five mortality  ${}_5q_0 = 0.1$ ; and the value of  $k = 0.5$ , which is given by the example. Since  $k$  is fixed, the researcher can match the log-quadratic model to the observed input  ${}_1m_0 = 0.05$ , finding the optimal value of  ${}_5q_0$  alone.

Step 2: recover the estimated value of  ${}_1m_0({}_5q_0, k)$ , that is implicit in that mortality schedule, dividing the estimated number of infant deaths  $q(1 \text{ year}; {}_5q_0, k)$  by the estimated number of person-years lived from 0 to 1 (i.e., using Equation 5). This step requires the value of  ${}_1L_0({}_5q_0, k)$ , which is the result of adding all  ${}_nL_x$  of the first year of life. Researcher can calculate the estimated values of  ${}_nL_x({}_5q_0, k)$  using Equation 4, under the assumption of constant force of mortality within the age interval.

Step 3: calculate the relative error of this approximation  $R_1({}_5q_0, k)$ , using the observed and the estimated values of  ${}_1m_0$ , as described by Equation 7.1.

Step 4: calculate the partial derivative of the error of approximation, to the form:  $\frac{\partial R_1(\cdot)}{\partial \ln[{}_5q_0]} =$

$\frac{R_1({}_5q_0 \cdot e^\Delta, k) - R_1({}_5q_0, k)}{\Delta}$ . Partial derivatives can be calculated numerically, multiplying the value of the parameter  ${}_5q_0$  by the factor  $\cdot e^\Delta$ , with a small value of  $\Delta = 10^{-5}$ ; and repeating steps 1, 2, and 3 to estimate the resulting error of approximation  $R_1({}_5q_0 \cdot e^\Delta, k)$ .

Step 5: update the value of  ${}_5q_0$ , using the Newton's method—in logs to prevent negative values—as defined by Equation 8. Since the researcher is only matching one input, Equation 8 in the paper can be simplified to be:

$$\ln[{}_5q_0^*] = \ln[{}_5q_0] - \left[ \frac{\partial R_1(\cdot)}{\partial \ln[{}_5q_0]} \right]^{-1} \cdot R_1({}_5q_0, k). \quad (\text{A1})$$

As indicated by Equation A1, the log of the updated value of  ${}_5q_0^*$  is equal to the log of its current value, minus the current error of approximation multiplied by the inverse of its partial derivative.

Step 6: iterate Equation A1, repeating steps 1, 2, 3, 4, and 5 until convergence is reached (e.g., when the absolute value of  $R_1({}_5q_0, k)$  is a very small quantity). As shown in Table A1, after three iterations the

absolute value of  $R_1({}_5q_0, k)$  is less than  $10^{-5}$ ; and the log-quadratic model is matching the inputs of this example.

<i>Worked example : <math>{}_1m_0 = 0.05</math> and <math>k = 0.5</math></i>					
<i>iteration</i>	${}_5q_0$	$k$	${}_1m_0({}_5q_0, k)$	$R_1({}_5q_0, k)$	$\partial R_1(\cdot)/\partial \ln[{}_5q_0]$
1	0.1000	0.5000	0.0784	-0.44994	-0.9661
2	0.0628	0.5000	0.0500	0.00097	-0.9711
3	0.0628	0.5000	0.0500	0.00000	-0.9711

**Table A1:** Worked example 1

Step 7: estimate the values of  ${}_1a_0$  and  ${}_4a_1$ , using Equation 3. Given the optimal values of  ${}_5q_0$  and  $k$ —reported in Table A1, the average age of infant deaths is estimated to be 0.2243 years; and the average number of years lived—in the age interval—by those dying from 1 to 5 is estimated to be 1.4339 years. The R-package associated with the log-quadratic model (<https://github.com/verhulsta/logquad5q0>) was updated to provide the values of  ${}_1a_0$  and  ${}_4a_1$ , given the inputs provided by the users.

## **2. The value of ${}_1m_0 = 0.025$ is given and 75 per cent of infant deaths have less than three months of life (i.e., $z(3m) = 0.75$ )**

Step 1: predict a mortality schedule for a feasible combination of parameters  $q(x; {}_5q_0, k)$ , using the Equation 2. As an initial guess, we have assumed an arbitrary level of under-five mortality  ${}_5q_0 = 0.05$ ; and the value of  $k = 0$ , which is representing a neutral age pattern of under-five mortality. No matter how realistic this initial guess could be, these two parameters are adjusted by successive approximations.

Step 2: recover the estimated value of  ${}_1m_0({}_5q_0, k)$ , that is implicit in that mortality schedule, dividing the estimated number of infant deaths  $q(12m; {}_5q_0, k)$  by the estimated number of person-years lived from 0 to 1  ${}_1L_0({}_5q_0, k)$ . The former is the sum of all  ${}_nL_x$  values corresponding to the first year of life.  ${}_nL_x({}_5q_0, k)$  can be calculated using Equation 4, under the assumption of constant force of mortality within the age interval.

Step 3: calculate the relative error of the first approximation  $R_1({}_5q_0, k)$ , using the observed and the estimated values of  ${}_1m_0$ , as described by Equation 7.1.

Step 4: recover the estimated value of  $z(3m; {}_5q_0, k)$ , using the values of  $q(3m; {}_5q_0, k)$  and  $q(1 \text{ year}; {}_5q_0, k)$  in Equation 6.

Step 5: calculate the relative error of the second approximation  $R_2({}_5q_0, k)$ , using the observed and the estimated values of  $z(3m)$ , as described by Equation 7.2.

Step 6: calculate the partial derivatives for each error of approximation for each input  $j$ , to the form:

$\frac{\partial R_j(\cdot)}{\partial \ln[{}_5q_0]} = \frac{R_j({}_5q_0 \cdot e^\Delta, k) - R_j({}_5q_0, k)}{\Delta}$ ; and  $\frac{\partial R_j(\cdot)}{\partial k} = \frac{R_j({}_5q_0, k + \Delta) - R_j({}_5q_0, k)}{\Delta}$ . This is multiplying the value of the parameters  ${}_5q_0$  by a factor  $e^\Delta$ , or increasing  $k$  by a small quantity  $\Delta = 10^{-5}$ ; and repeating the corresponding steps to estimate the resulting errors of approximation:  $R_j({}_5q_0 \cdot e^\Delta, k)$  or  $R_j({}_5q_0, k + \Delta)$ .

Step 7: update the values of  ${}_5q_0$  and  $k$ , using the Newton's method. Hence, the log of the updated values of  ${}_5q_0^*$  and  $k$  are equal their current value, minus the inverse of the Jacobian matrix—of first-order partial derivatives—multiplied by a column vector of current errors of approximation.

Step 8: iterate Equation 8 in the paper, repeating steps 1, 2, 3, 4, 5, 6, and 7 until convergence is reached (e.g., when the absolute value of  $R_1({}_5q_0, k)$  and  $R_2({}_5q_0, k)$  are very small quantities). If the Jacobian matrix is near to be a singular matrix (i.e., the determinant is close to zero), its inverse can be calculated as a Moore-Penrose pseudoinverse. As reported in Table A2, after three iterations the absolute value of the approximation errors  $R_1({}_5q_0, k)$  and  $R_2({}_5q_0, k)$  are less than  $10^{-5}$ .

Step 9: estimate the values of  ${}_1a_0$  and  ${}_4a_1$ , using the optimal values of  ${}_5q_0$  and  $k$  in Equation 3. Given the two inputs of this worked example, the average age of infant deaths is estimated to be 0.1645 years; and the average number of years lived—in the age interval—by those dying from 1 to 5 is estimated to be 1.5433 years.

<i>Worked example : <math>{}_1m_0 = 0.025</math> and <math>z(3m) = 0.75</math></i>							
<i>iteration</i>	${}_5q_0$	$k$	${}_1m_0({}_5q_0, k)$	$R_1({}_5q_0, k)$	$z(3m; {}_5q_0, k)$	$R_2({}_5q_0, k)$	$\det[J]$
1	0.0500	0.0000	0.0419	-0.51583	0.7267	0.03152	-0.1572
2	0.0301	0.2269	0.0249	0.00220	0.7427	0.00985	-0.1545
3	0.0300	0.1627	0.0250	-0.00000	0.7500	-0.00000	-0.1545

$\det[J]$  is the determinant of the Jacobian matrix [first derivatives of  $R_j({}_5q_0, k)$ ].

**Table A2:** Worked example 2

## Appendix 2: Bayesian evaluation of the method

Considering the average age of infant deaths is a small positive value—that has changed over time—and some small errors of prediction might imply large proportions of the observed value, we evaluate the accuracy and precision of our indirect approach by means of the relative error of predicting  ${}_1a_0$ , as indicated by Equation A2. The function  ${}_1a_0({}_5q_{0,i}, k_i)$ , denotes an estimated value of the average age of infant deaths—at a given values of  ${}_5q_{0,i}$  and  $k_i$ , corresponding to the level and the age pattern of under-five mortality of a particular of a country-year  $i$ . This function was explicitly defined by Equation 3 of the paper. Note that the prediction of  ${}_4a_1$  can be evaluated by a similar expression.

$$\ln[{}_1a_{0,i}^{Observed}] - \ln[{}_1a_0({}_5q_{0,i}, k_i)] = \mu + \epsilon_i. \quad (A2)$$

We assume that the relative error of prediction—or model residual—has a normal distribution (N) with mean  $\mu$  and variance  $1/\varphi$ . The constant  $\mu$  is informing the bias—or lack of accuracy—and the estimated value of  $\varphi$  is measuring the precision of the method, when the log-quadratic model is fitted to the  $M = 1,219$  life tables used for evaluation. This normal distribution corresponds to the likelihood of the method, when the values of  ${}_5q_0$  and  $k$  are the result of matching the log-quadratic model to one or two observed inputs (e.g., observed central death rates, the proportion of infant deaths below certain age, etc.).

We then estimate  $\mu$  and  $\varphi$  by means of conjugate prior distributions. From this perspective, the bias coefficient  $\mu$ , is also assumed to have a normal prior with mean zero and variance hyperparameter  $1/\psi$  (i.e., the same parametric distribution of the likelihood); and the precision coefficient  $\varphi$ , is assumed to have a gamma distribution ( $\Gamma$ ), with shape hyperparameter  $\alpha$ , and inverse scale hyperparameter  $\beta$ . The advantage of the conjugacy is to get the same parametric function for the posterior distribution of each coefficient (Gelman et al. 2014). Hence, given the parametric assumptions, the bias coefficient  $\mu$  has a normal posterior distribution described by Equation A3; and the precision coefficient  $\varphi$ , has a gamma distribution denoted by Equation A4. Note these two distributions are conditional to each other.

$$\mu | {}_1a_0, \varphi \sim N\left(\frac{\varphi \cdot \sum_{i=1}^M \ln[{}_1a_{0,i}^{Observed}] - \ln[{}_1a_0({}_5q_{0,i}, k_i)]}{\psi + M \cdot \varphi}, \frac{1}{\psi + M \cdot \varphi}\right), \quad (A3)$$

$$\varphi | {}_1a_0, \mu \sim \Gamma\left(\alpha + \frac{M}{2}, \beta + \frac{\sum_{i=1}^M \epsilon_i^2}{2}\right). \quad (\text{A4})$$

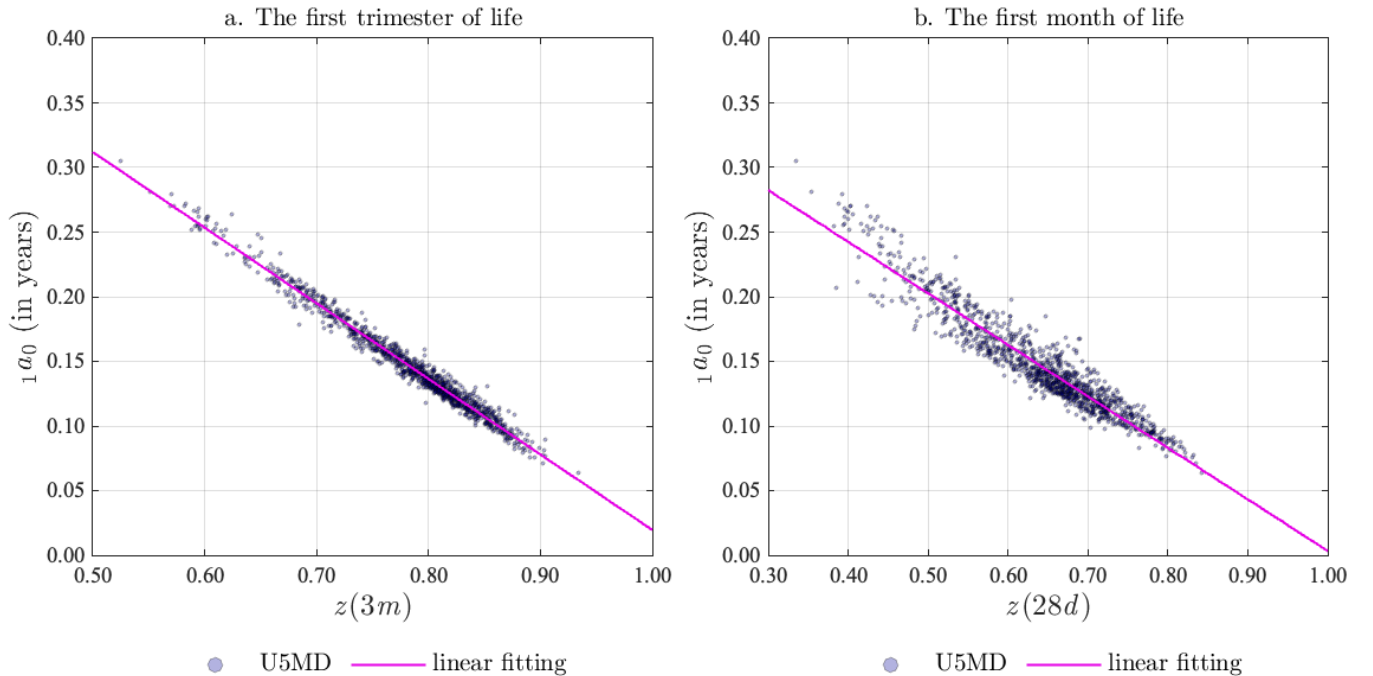
We draw samples from the conditional posterior distributions of  $\mu$  and  $\varphi$  by means of a Markov chain simulation, using the *Gibbs sampler* algorithm—alternating and updating both Equation A3 and Equation A4 after each iteration. The Markov chain was iterated 10,000 times with an initial burn-in period of 2,500 additional iterations. The Gibbs sampler is an adequate choice, considering the reduced number of parameters to be estimated and the use of conjugate priors. More sophisticated Markov Chain Monte Carlo methods—drawing the prior and then approximating the posterior distributions—can be implemented (e.g., Metropolis-Hasting or Hamiltonian Monte Carlo) at a higher computational cost. For simplicity, priors are assumed to be:  $\psi = 10^{-4}$ ,  $\alpha = 10^{-2}$ , and  $\beta = 10^{-2}$ ; resulting in flat distributions with large variance.

$$\text{RMSE} = \sqrt{\mu^2 + 1/\varphi}. \quad (\text{A5})$$

Values of  $\mu$  and  $\varphi$  can be used to calculate the Root Mean Square Error (RMSE)—by means of Equation A5. For each drawn pair of  $\mu$  and  $\varphi$ , we estimate the RMSE to approximate its posterior distribution and uncertainty bounds. Using the same evaluation set of 1,219 life tables and the same prior distributions, we estimate the accuracy and precision for each sex, for both sexes combined, and for all previous formulae to approximate the values of  ${}_1a_0$  and  ${}_4a_1$ .

### Appendix 3: Direct estimation of ${}_1a_0$ , as a competing approach

As a robustness test, we want to compare our indirect method of estimating  ${}_1a_0$  and  ${}_4a_1$  with a direct—and computationally less expensive—alternative that is using the best predictors of the age patterns of under-five mortality and taking advantage of the granularity of the U5MD. Following the traditional approaches, we fit two equations regressing the value of  ${}_1a_0$  on the proportion of infant deaths below certain age  $x$ — $z(x)$ , either 28 days or 3 months. The average age of infant deaths is a metric of the age distribution of deaths from 0 to 1, hence the strong correlation of  ${}_1a_0$  and the proportion of infant deaths during the first trimester or the first month of life—described in Figure A1.



**Figure A1:** The average age of infant deaths  ${}_1a_0$ , and the proportion of infant deaths during the first months of life (both sexes combined)

Both  ${}_1a_0$  and  $z(x)$  are related to the age pattern of mortality in early ages, but the proportion of infant deaths below certain age  $x$   $z(x)$ , has the advantage of being directly observed and only requires few

operations.  $z(x)$  bears relevant information for the estimation of  ${}_1a_0$ , either by means of the log-quadratic model—as an indirect method—or a linear regression as Figure A1 is suggesting.

$${}_1a_{0,i}^{Observed} = \delta_0 + \delta_1 \cdot z_i(x) + e_i. \quad (A6)$$

To predict the average age of infant deaths, researchers have the option to use a simple equation—regressing the average age of infant deaths on  $z(3m)$  or  $z(28d)$ —as indicated by Equation A6 for any country-year  $i$ . Compared to similar direct approaches, this is the first attempt to define a simple equation depending on one input that is actually related to the age distribution of deaths during the first year of life. Resulting coefficients for each sex and for both sexes combined are reported in Table A3.

	Direct estimation, using $z(3m)$			Direct estimation, using $z(28d)$		
	Female	Male	Both sexes	Female	Male	Both sexes
$\delta_0$	+0.6027 [+0.5973, +0.6075]	+0.5981 [+0.5928, +0.6028]	+0.6043 [+0.5995, +0.6085]	+0.4084 [+0.4029, +0.4141]	+0.3954 [+0.3902, +0.4008]	+0.4018 [+0.3967, +0.4070]
$\delta_1 z(3m)$	-0.5815 [-0.5877, -0.5747]	-0.5779 [-0.5838, -0.5712]	-0.5848 [-0.5902, -0.5787]			
$\delta_1 z(28d)$				-0.4052 [-0.4138, -0.3967]	-0.3913 [-0.3995, -0.3833]	-0.3985 [-0.4064, -0.3907]

p50/[p2.5, p97.5]

**Table A3:** Regression coefficients for the direct estimation of the average age of infant deaths

The mortality during the first month of life has been more relevant and available, but there are methodological reasons to prefer  $z(3m)$  over  $z(28d)$ . If  $z(3m)$  is used as a predictor of the average age of infant deaths, then the most traditional inputs— ${}_1q_0$  and  ${}_1m_0$ —are not statistically significant (i.e., Equation A5 does not require additional inputs). In addition to the strong correlation,  $z(3m)$  has two relevant advantages in demographic estimation. This proportion is not affected by the historical change in the definition of the neonatal mortality from one calendar month of about 30 days to an analytical month of 28 days (Gourbin and Masuy-Stroobant 1995).  $z(3m)$  is less sensitive to the imprecision in the reported ages at death—when life tables are calculated from retrospective data. Particularly, approximation errors around the first month of life when the level of mortality is higher.

#### Appendix 4: Sensitivity of other life table indicators to the values of ${}_1a_0$ and ${}_4a_1$

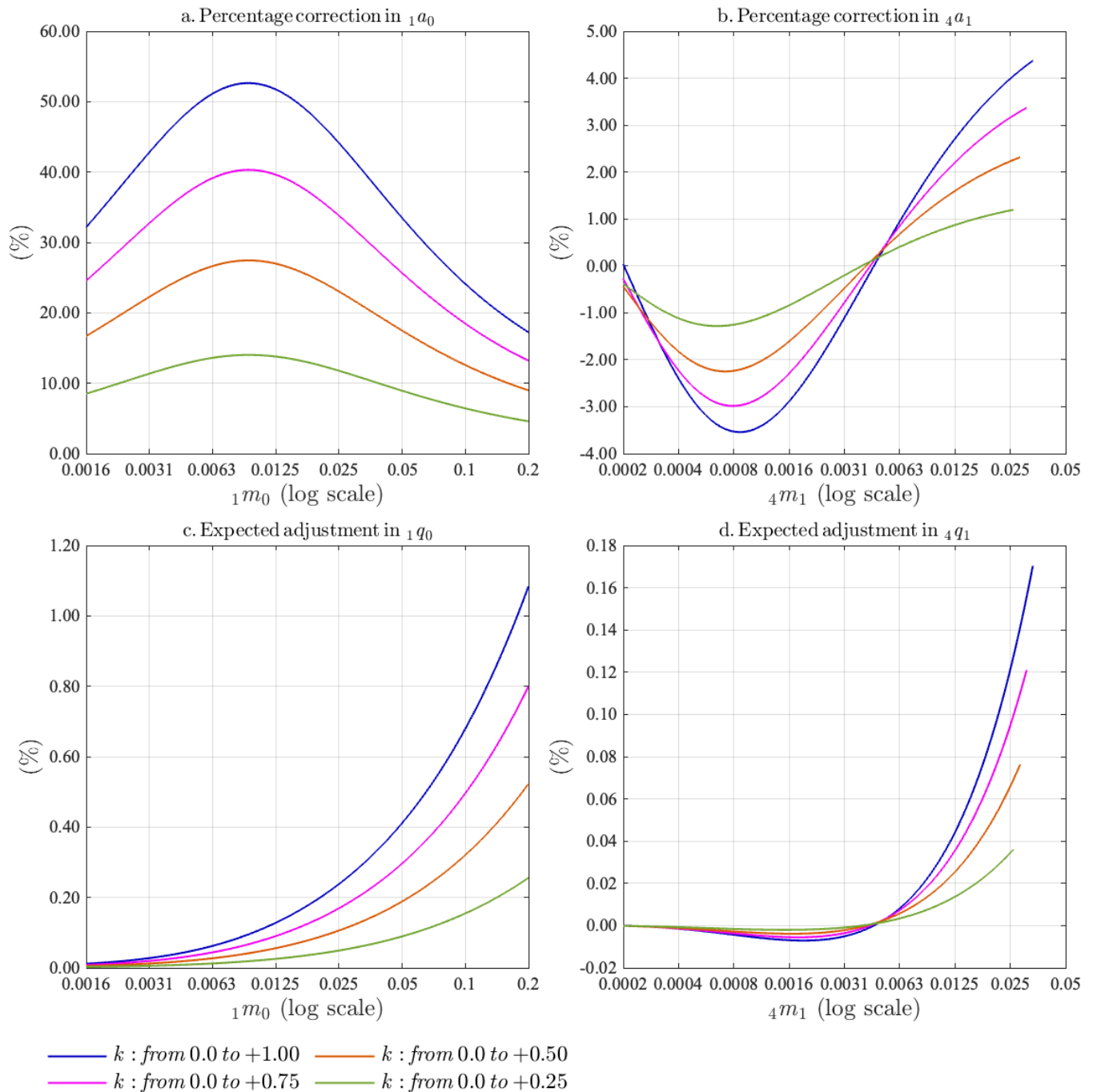
Correcting the values of  ${}_1a_0$  and  ${}_4a_1$  implies some other adjustments. The sensitivity of this correction can be quantified as the  ${}_na_x$ -elasticity of a given life table function. Some of these functions may depend directly on the value of  ${}_1a_0$  or  ${}_4a_1$ —such as the infant or the under-five mortality rate, or indirectly—though the adjusted value of these probabilities of dying. For example, elasticities can be calculated by taking the partial derivative—and rearranging terms—of key equations for life table construction, such as the Keyfitz's formula to calculate  ${}_nq_x$  as a function of  ${}_nm_x$  and  ${}_na_x$ . The resulting  ${}_na_x$ -elasticity of  ${}_nq_x$  measures the percentual response in  ${}_nq_x$  to a percentual increase in  ${}_na_x$ , whose solution is indicated by Equation A7.

$$\frac{\partial {}_nq_x / {}_nq_x}{\partial {}_na_x / {}_na_x} = {}_nq_x \cdot \frac{{}_na_x}{n}. \quad (A7)$$

Since this elasticity is the product of two positive numbers that are less than one,  ${}_nq_x$  is expected to increase less than proportionally to the increase in  ${}_na_x$ . Larger adjustments are expected in populations with higher levels of  ${}_nq_x$ ; where deaths are less concentrated at the beginning of the age interval (i.e., larger  ${}_na_x$ ); or after a substantial correction on  ${}_na_x$ . In the practice, this correction would depend on how far  ${}_na_x$  is from its actual value. To make a *ceteris paribus* assessment, we quantify the change in the predicted value of  ${}_na_x$ , when the parameter  $k = 0$  (i.e., the central tendency of the log-quadratic model) is replaced by its

optimal value, fitting the log-quadratic model to a specific age pattern of under-five mortality and keeping constant the level of  ${}_n m_x$ .

Panel A and Panel B of Figure A2 show the percentage correction in  ${}_1 a_0$  and  ${}_4 a_1$ , when the parameter  $k$  changes from zero to some arbitrary but feasible values. As described by Panel A, in the extreme case of  $k = 1.0$ ,  ${}_1 a_0$  would be corrected up to 52.63 per cent and the correction of a moderated scenario of  $k = 0.5$  would be up to 27.44 per cent, keeping constant the value of  ${}_1 m_0$ . Indeed, the magnitude of the correction depends on the level of mortality. As described by Panel B,  ${}_4 a_1$  is less sensitive to the same change in the value of  $k$  and its resulting value can be corrected upwards or downwards, depending on the level of  ${}_4 m_1$ . Corrections range from -3.54 to 4.37 percent. Similar corrections can be calculated—in the opposite direction—if the parameter  $k$  is assumed to change from zero to some negative values.



**Figure A2:** Percentage correction in  ${}_1 a_0$  and  ${}_4 a_1$ ; and expected adjustment in  ${}_1 q_0$  and  ${}_4 q_1$  (both sexes combined)

Panel C and Panel D of Figure A2 show the expected adjustment in  ${}_1q_0$  and  ${}_4q_1$ , given the same changes in the parameter  $k$ . These adjustments are the product of each  ${}_na_x$ -elasticity of  ${}_nq_x$  and the corrected percentage in  ${}_na_x$ . As described by Panel C, the expected adjustments in the infant mortality rate  ${}_1q_0$ , are small and more relevant for populations with higher levels of mortality. In the extreme case of  $k = 1.0$ , the adjustment is not expected to be more than 1.08 per cent. The Panel D shows that the expected adjustments in the probability of dying during the childhood  ${}_4q_1$ , are negligible. Even in the range of higher levels of  ${}_4m_1$ , the adjustments are not expected to be greater than 0.17 percent.

Adjustments in  ${}_1q_0$  and  ${}_4q_1$  can be very small, given a reasonable correction in the value of  ${}_1a_0$  and  ${}_4a_1$ . However, there are two reasons to consider these changes as important considerations for life table construction. On the one hand, these adjustments propagate to other life table functions. On the other hand, when the life table is used for projecting a population via survivorship ratios, even a small adjustment in the value of  ${}_1q_0$  can produce undercounts/overcounts of some thousands, which is not the ideal case when projecting large populations or for extended periods of time. As denoted by Equation A8, other life table functions that are linearly depending on  ${}_1q_0$  have the same elasticity. Those are the number of infant deaths in a life table, the estimated number of person-years from 0 to 1, and the projected number of people with less than one year.

$$\frac{\partial {}_1q_0 / {}_1q_0}{\partial {}_1a_0 / {}_1a_0} = \frac{\partial {}_1d_0 / {}_1d_0}{\partial {}_1a_0 / {}_1a_0} = \frac{\partial {}_1L_0 / {}_1L_0}{\partial {}_1a_0 / {}_1a_0} = \frac{\partial {}_1N_0 / {}_1N_0}{\partial {}_1a_0 / {}_1a_0} = {}_1q_0 \cdot {}_1a_0. \quad (\text{A8})$$

Using Equation A7, similar elasticities can be calculated for the probability of dying during childhood, the number of deaths from 1 to 5 in a life table, and the estimated number of person-years lived within the same age interval.

Correcting the value of  ${}_1a_0$  will also implies a small variation in the number of years lived above the age 0 and the resulting life expectancy at birth, indicated by  $T_0$  and  $e_0$ , respectively. However, as suggested by Equation A9, the sign of this elasticity is not to be assumed strictly positive or negative, inasmuch as it depends on the level and age pattern of mortality (i.e., if  ${}_1q_0$  is greater/less than  ${}_1L_0/T_0$ ).

$$\frac{\partial T_0 / T_0}{\partial {}_1a_0 / {}_1a_0} = \frac{\partial e_0 / e_0}{\partial {}_1a_0 / {}_1a_0} = - \left[ {}_1q_0 - \frac{{}_1L_0}{T_0} \right] \cdot \frac{{}_1q_0}{1 - {}_1q_0} \cdot {}_1a_0. \quad (\text{A9})$$

Some other life table functions above the age 1 would be adjusted, after correcting the average age of infant deaths. As indicated by Equation A10, when the value  ${}_1a_0$  is corrected upwards, there is a small adjustment downwards in the survivorship function, the number of deaths, the number of person-years lived within the age interval, and the number of years lived above the age  $x + 1$ , for any  $x$  greater than zero. Note, however, that the adjustment is smaller than the one expected in  ${}_1q_0$ , inasmuch as this elasticity is the product of three numbers that are less than one in absolute value.

$$\frac{\partial l(x+1) / l(x+1)}{\partial {}_1a_0 / {}_1a_0} = \frac{\partial n d_{x+1} / n d_{x+1}}{\partial {}_1a_0 / {}_1a_0} = \frac{\partial n L_{x+1} / n L_{x+1}}{\partial {}_1a_0 / {}_1a_0} = \frac{\partial T_{x+1} / T_{x+1}}{\partial {}_1a_0 / {}_1a_0} = - \frac{{}_1q_0}{1 - {}_1q_0} \cdot {}_1q_0 \cdot {}_1a_0. \quad (\text{A10})$$

The projected number of people at the age 1  ${}_1N_1$ , is usually calculated from the current population at age 0  ${}_1N_0$ , and the survivorship ratio  ${}_1L_1 / {}_1L_0$ . When the average age of infant deaths is corrected upwards, we would expect the denominator of this ratio to increase and the numerator to increase, as indicated by Equation A8 and Equation A10, respectively. Hence, the  ${}_1a_0$ -elasticity of  ${}_1N_1$  for the first year of projections is negative, as denoted by Equation A11. This elasticity is—in absolute value—greater than the one corresponding to the infant mortality rate.

$$\frac{\partial {}_1N_1 / {}_1N_1}{\partial {}_1a_0 / {}_1a_0} = - \frac{{}_1q_0}{1 - {}_1q_0} \cdot {}_1a_0. \quad (\text{A11})$$

Note, however, that if  ${}_1N_1$  is calculated from the projected population at age 0  ${}_1N_0$ , the latter is already adjusted by the corrected value of  ${}_1a_0$ , the effect through  ${}_1L_0$  cancels out, and the  ${}_1a_0$ -elasticity of  ${}_1N_1$  is the same elasticity of  ${}_1L_1$  for a subsequent year of projections.

Finally, there are some life table indicators that are inelastic—and are expected to remain the same. Probabilities of dying and life expectancies above the age 1 and the projected number of people with two or more years of life—for the first year of projections—would not be adjusted as a result of a correction on the average age of infant deaths, as defined by Equation A12.

$$\frac{\partial nq_{x+1}/nq_{x+1}}{\partial {}_1a_0/{}_1a_0} = \frac{\partial e_{x+1}/e_{x+1}}{\partial {}_1a_0/{}_1a_0} = \frac{\partial nN_{x+2}/nN_{x+2}}{\partial {}_1a_0/{}_1a_0} = 0. \quad (\text{A12})$$

## References

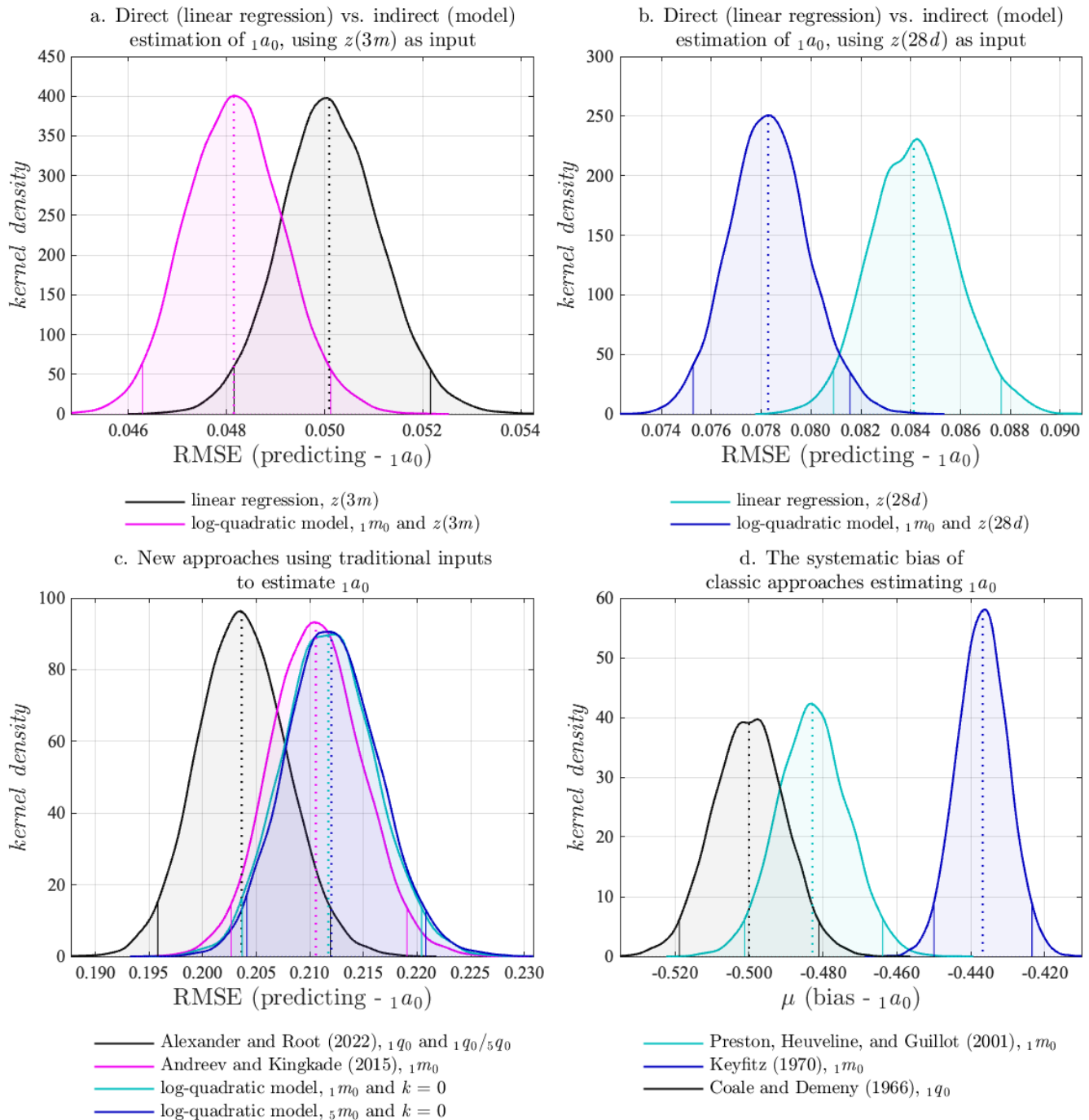
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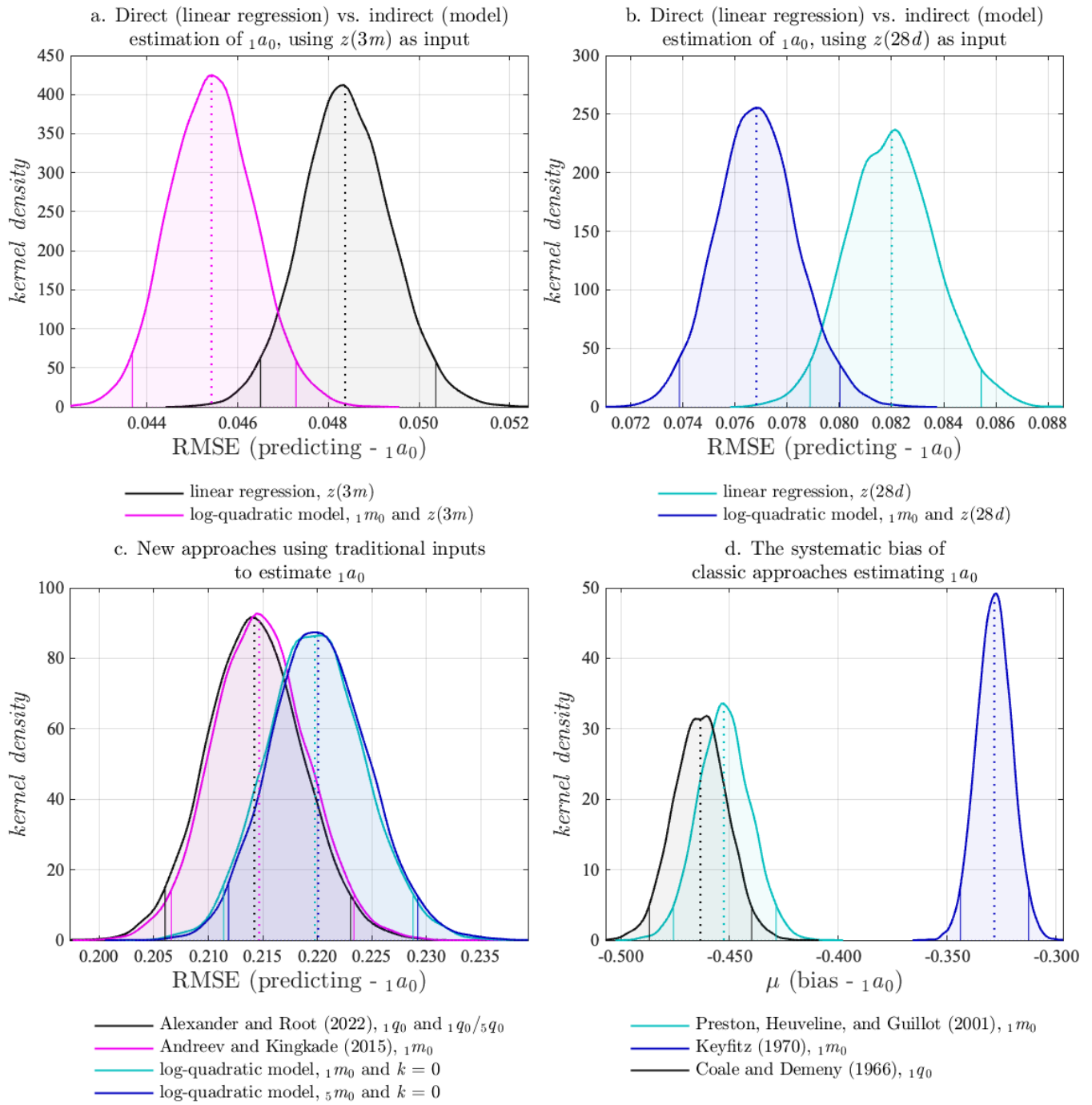


## Appendix 5: Additional figures

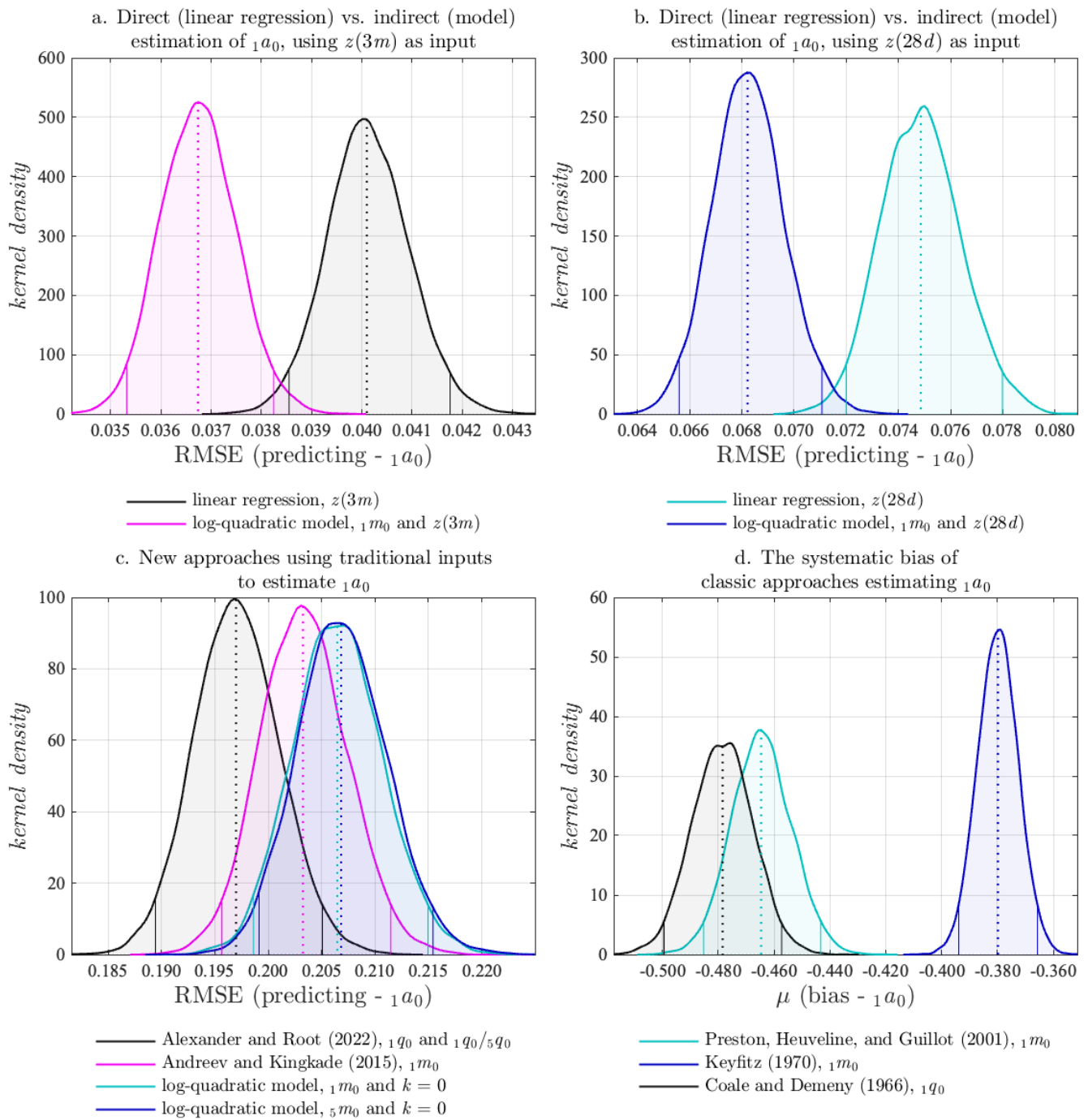
The main results of evaluating the performance of our method are described by Figure A3 (female population), Figure A4 (male population), and Figure A5 (both sexes combined). Gaussian kernels were used to smooth the posterior distributions of the estimated bias, the precision, and the resulting Root Mean Square Error (RMSE). Panel A and panel B show the RMSE of our proposed indirect estimation compared to a direct counterpart, using the same leading predictor. Panel C shows the RMSE of the new approaches compared to our indirect estimation, using similar inputs, and not taking advantage of the age distributions of deaths during the first year of life. Finally, panel D shows the systematic bias of the classic approaches in estimating  ${}_1a_0$ , given the observed values of the *Under-5 Mortality Database*.



**Figure A3:** The main results of predicting the average age of infant deaths  ${}_1a_0$ , using the *Under-5 Mortality Database* (female)

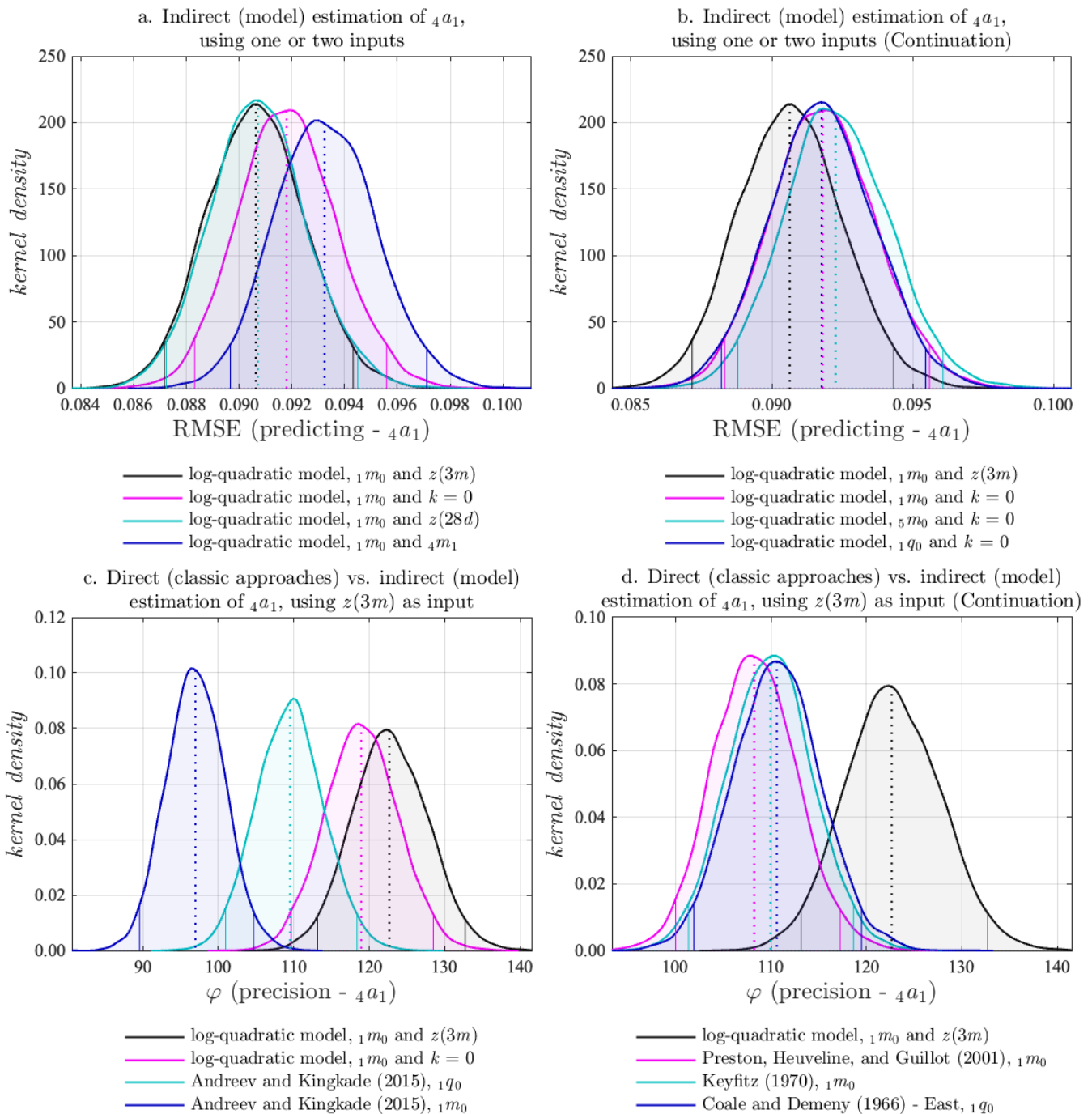


**Figure A4:** The main results of predicting the average age of infant deaths  ${}_1a_0$ , using the *Under-5 Mortality Database (male)*

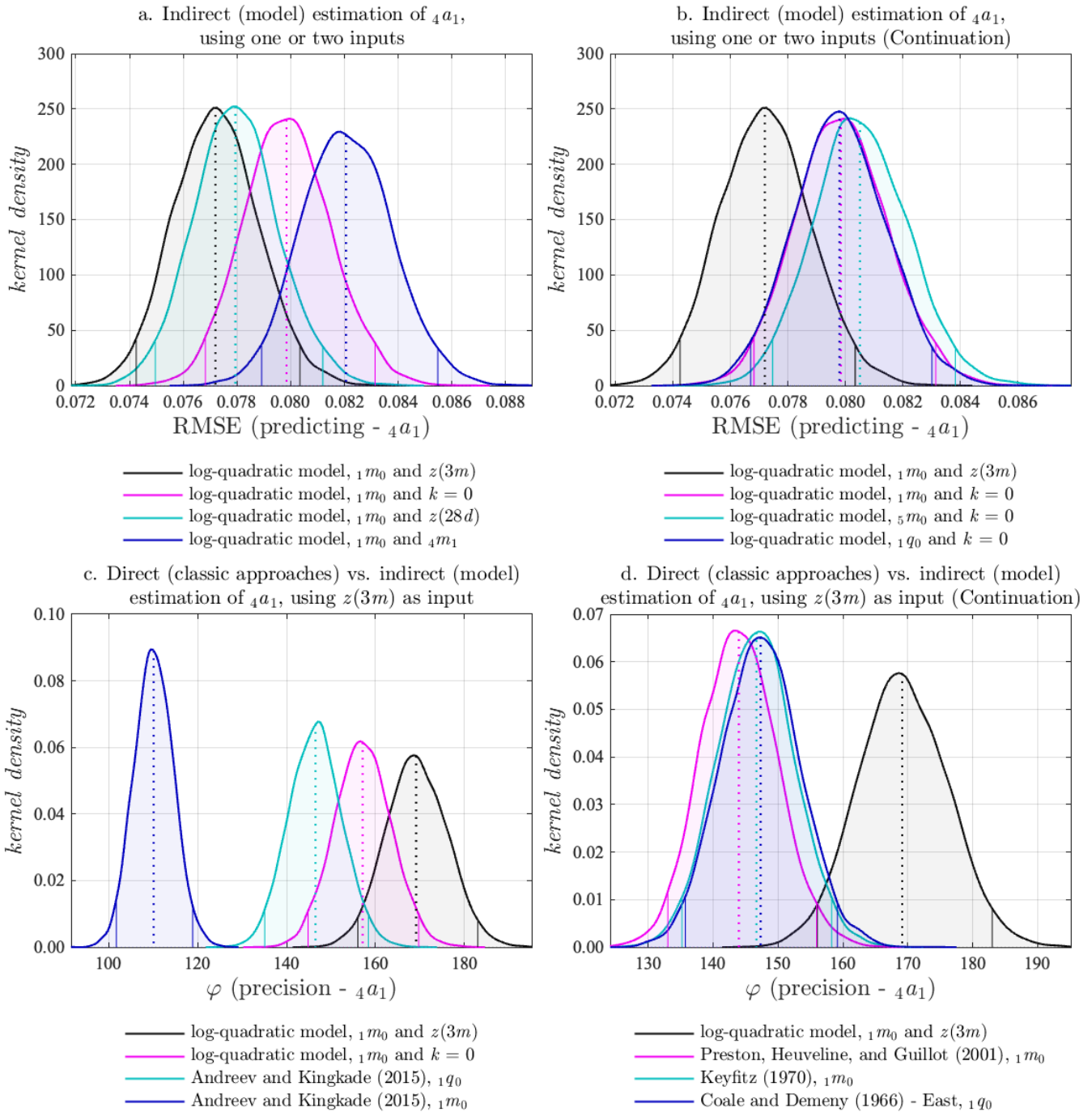


**Figure A5:** The main results of predicting the average age of infant deaths  ${}_1a_0$ , using the *Under-5 Mortality Database* (both sexes combined)

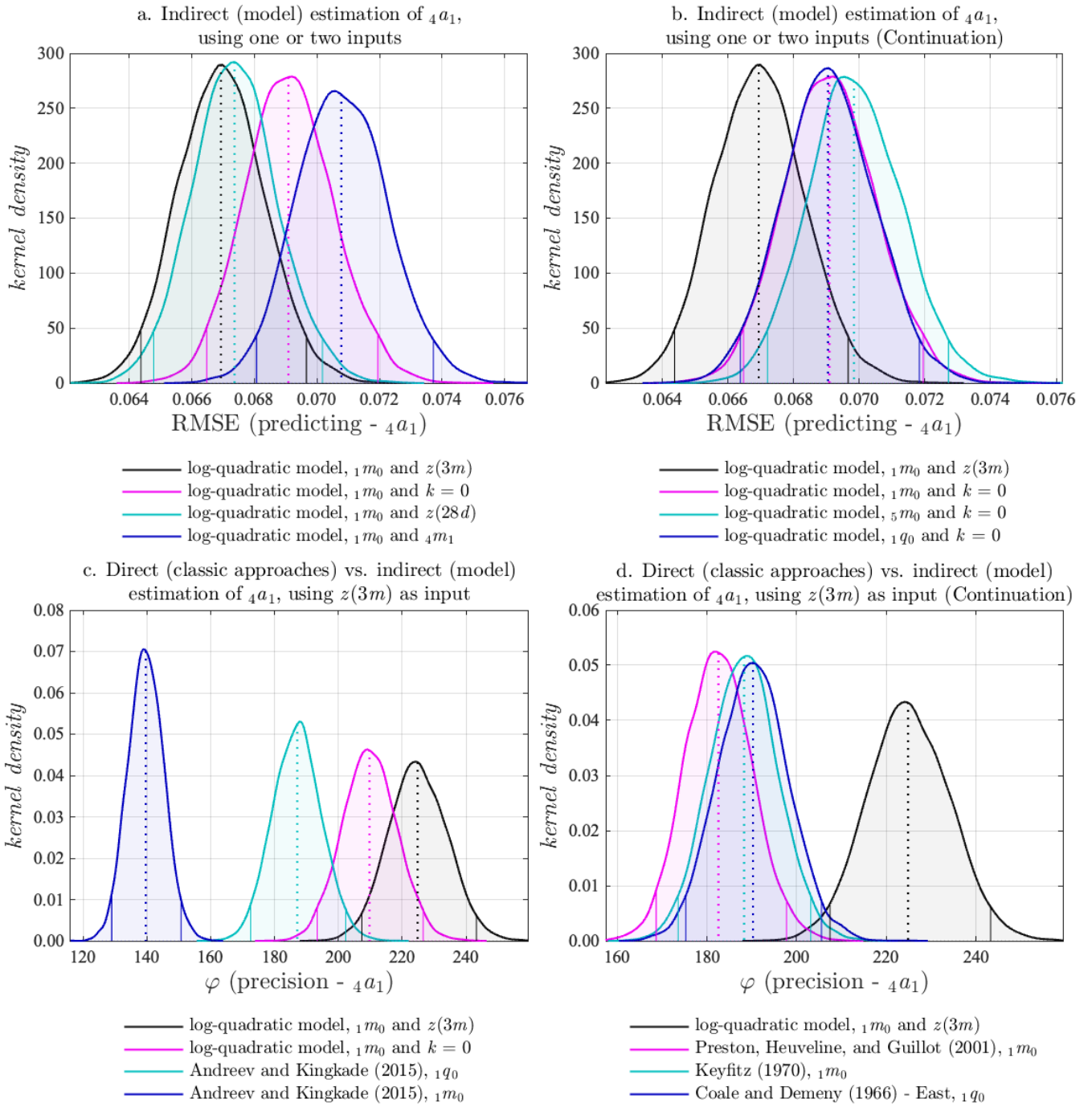
Figure A6 (female population), Figure A7 (male population), and Figure A8 (both sexes combined) describe the performance of log-quadratic in estimating the average number of years lived–in the age interval–by those dying from 1 to 5  ${}_4a_1$ . These figures compare the predictive power of using different inputs, contrasting the fitting of the log-quadratic model and the classic methods. The panel A shows the RMSE of fitting the log-quadratic model and the moderated improvement of using the inputs that best inform the age patterns of under-five mortality. The panel B contrasts the predictive power of using only one input with the best combination of two inputs. The panel C shows that a log-quadratic model–matching two inputs and informing the correct pattern of under-five mortality–is significantly more precise than the equation adapted by Preston, Heuveline, and Guillot (2001) and the general rule assumed by Keyfitz and Flieger (1971). Finally, the panel D reassures the same result, contrasting the precision of the log-quadratic model with the other three regional families that are part of the Coale and Demeny’s (1966) methodology.



**Figure A6:** The main results of predicting the average number of years lived–in the age interval–by those dying from 1 to 5  ${}_4a_1$ , using the *Under-5 Mortality Database (female)*



**Figure A7:** The main results of predicting the average number of years lived—in the age interval—by those dying from 1 to 5  ${}_4a_1$ , using the *Under-5 Mortality Database (male)*



**Figure A8:** The main results of predicting the average number of years lived—in the age interval—by those dying from 1 to 5  ${}_4a_1$ , using the *Under-5 Mortality Database* (both sexes combined)