Web-based Supplementary Materials for “A log-rank-type test to compare net survival distributions” by

Nathalie Grafféo¹-², Fabienne Castell³, Aurélien Belot⁴-⁸, and Roch Giorgi¹,²,⁹,*

¹INSERM, UMR912 “Sciences Économiques et Sociales de la Santé et Traitement de l’Information Médicale” (SESSTIM), F-13006, Marseille, France
²Université d’Aix-Marseille, UMR S912, IRD, F-13006, Marseille, France
³Université d’Aix-Marseille, CNRS, Centrale Marseille, I2M, UMR 7373, F-13453, Marseille, France
⁴Service de Biostatistique, Hospices Civils de Lyon, F-69003, Lyon, France
⁵Université de Lyon, F-69000, Lyon, France
⁶Université Lyon 1, F-69100, Villeurbanne, France
⁷CNRS, UMR5558, Laboratoire de Biométrie et Biologie Évolutive, Équipe Biostatistique-Santé, F-69100, Villeurbanne, France
⁸Institut de Veille Sanitaire, DMCT, Saint-Maurice, France
⁹APHM, Hôpital Timone, BIOSTIC, Marseille, France

*email: roch.giorgi@univ-amu.fr

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Web Appendix A. Proof of \( E \langle Z^w_h \rangle < \infty \)

We have \( \langle Z^w_h \rangle (T) = \sum_{i=1}^{k} \left\{ \int_{0}^{T} 1(Y^w_i(s) > 0) \left( \delta_{hl} - \frac{Y^w_h(s)}{Y^w_i(s)} \right) \frac{n_i}{\left( \tilde{S}_{P,l,i}(s) \right)^2} \right\} \).

Note that \( \forall s \in [0, T] \ 1(Y^w_i(s) > 0) \left( \delta_{hl} - \frac{Y^w_h(s)}{Y^w_i(s)} \right)^2 \leq 1 \). Thus,

\[
E \langle Z^w_h \rangle (T) \leq \sum_{i=1}^{k} n_i E \left\{ \int_{0}^{T} \frac{Y_{i,1}(s)}{\tilde{S}_{P,l,1}(s)} \left( \tilde{\lambda}_{P,l,1}(s) + \lambda_E(s) \right) ds \right\}.
\]

As \( T_E, T_P \) and \( C \) are conditionally independent given \( X \), we can write

\[
E(Y_{i,1}(s) \mid X_{i,1}) = S_{C,l}(s) S_E(s) \tilde{S}_{P,l,1}(s).
\]

Using that \( 0 \leq S_{C,l}, S_E \leq 1 \), we get

\[
E \langle Z^w_h \rangle (T) \leq \sum_{i=1}^{k} n_i E \left\{ \int_{0}^{T} \frac{\tilde{\lambda}_{P,l,1}(s)}{S_{P,l,1}(s)} ds + \int_{0}^{T} \frac{S_E(s) \lambda_E(s)}{S_{P,l,1}(s)} ds \right\}.
\]

Note that \( \int_{0}^{T} \frac{\tilde{\lambda}_{P,l,1}(s)}{S_{P,l,1}(s)} ds = \int_{0}^{T} \tilde{\lambda}_{P,l,1}(s) e^{\tilde{\lambda}_{P,l,1}(s)} ds = \left[ e^{\tilde{\lambda}_{P,l,1}(s)} \right]_{0}^{T} = \frac{1}{S_{P,l,1}(T)} - 1 \).

Moreover, for \( 0 \leq s \leq T \), \( \tilde{S}_{P,l,1}(s) \geq \tilde{S}_{P,l,1}(T) \) and \( \int_{0}^{T} S_E(s) \lambda_E(s) ds = 1 - S_E(T) \), we get

\[
E \langle Z^w_h \rangle (T) \leq 2E \left( \sum_{i=1}^{k} \frac{n_i}{S_{P,l,1}(T)} \right),
\]

which is finite according to the second assumption in (2) in the main document.

Web Appendix B. Proof of the asymptotic distribution of the test statistic under the null

By the law of large numbers: \( \forall h \in [1; k], \frac{Y^w_h(s)}{n} \xrightarrow{(a.s.)} \alpha_h S_{C,h}(s) S_E(s). \)

Let us define

\[
y^w_i(s) := \sum_{h=1}^{k} \alpha_h S_{C,h}(s) S_E(s) \]

\[
a_h(s) := \frac{\alpha_h S_{C,h}(s)}{\sum_{i=1}^{k} \alpha_i S_{C,i}(s)}
\]

We introduce

\[
V_h = \sum_{i=1}^{k} \int_{0}^{T} 1(y^w_i(s) > 0)(\delta_{hl} - a_h(s)) \frac{n_i}{S_{P,l,i}(s)} dM_{i,1}(s).
\]
We can write under the null hypothesis $Z^w_h(T) = V_h + R^h_T$, where

$$R^h_T = \sum_{l=1}^k \sum_{i=1}^{n_l} \int_0^T \left\{ \mathds{1}(y^w_i(s) > 0) \left( \delta_{hl} - \frac{Y^w_h(s)}{Y^w_i(s)} \right) - \mathds{1}(y^w_i(s) > 0) (\delta_{hi} - a_h(s)) \right\} \frac{dM_{l,i}(s)}{S_{P,l,i}(s)}.$$

Let us denote

$$\Sigma^2_{hj}(T) = E \left\{ \sum_{l=1}^k \alpha_l \int_0^T \mathds{1}(y^w_i(s) > 0) (\delta_{hl} - a_h(s))(\delta_{jl} - a_j(s)) \frac{S_{C,l}(s)S_{E}(s)}{S_{P,l,i}(s)} \left( \tilde{\lambda}_{P,l,1}(s) + \lambda_E(s) \right) ds \right\}.$$

We are going to prove

Lemma 1: \( \frac{1}{\sqrt{n}} (V_1, \cdots, V_k) \xrightarrow{(d)} \mathcal{N}(0, \Sigma^2(T)) \), where \( \Sigma^2(T) \) is the matrix whose entries are the \( \Sigma^2_{hj}(T) \).

Lemma 2: Under the null hypothesis, \( \frac{1}{\sqrt{n}} R^h_T \xrightarrow{L^2} 0. \)

By Slutsky’s lemma, these two lemmas imply that under the null hypothesis,

$$\frac{1}{\sqrt{n}}(Z^w_1(T), \cdots, Z^w_k(T)) \xrightarrow{(d)} \mathcal{N}(0, \Sigma^2(T)).$$ (1)

Proof of Lemma 1.

Let us denote, for \((h, j) \in [1; k]^2\)

$$W_{h,j} := \sum_{i=1}^{n_j} \int_0^T \mathds{1}(y^w_i(s) > 0) (\delta_{hj} - a_h(s)) \frac{dM_{j,i}(s)}{S_{P,j,i}(s)}.$$

We can prove that vector

$$\frac{1}{\sqrt{n}} (W_{h,j})_{1 \leq h, j \leq k} \xrightarrow{(d)} \mathcal{N}(0, S^2(T)) \quad (*)$$

where \( S^2(T) \) is a \( k^2 \) square symmetric matrix whose entries are given by

$$S^2_{(hj), (h'j')} = \delta_{jj'} \alpha_j E \left\{ \int_0^T \mathds{1}(y^w_i(s) > 0) (\delta_{hj} - a_h(s))(\delta_{h'j} - a_{h'}(s)) \frac{S_{C,j}(s)S_{E}(s)}{S_{P,j,i}(s)} \left( \tilde{\lambda}_{P,j,1}(s) + \lambda_E(s) \right) ds \right\}.$$
Indeed, let \((\theta_{h,j})_{1 \leq h,j \leq k} \in \mathbb{R}^{k^2}\).

\[
E \left\{ \exp \left( \frac{i}{\sqrt{n}} \sum_{h,j} \theta_{h,j} W_{h,j} \right) \right\} = \prod_{j=1}^{k} E \left\{ \exp \left( \frac{i}{\sqrt{n}} \sum_{p=1}^{n_j} \int_0^T \mathbb{1}(y_u(s) > 0) \theta_{h,j} (\delta_{hj} - a_h(s)) \frac{dM_{j,p}(s)}{S_{P,j,p}(s)} \right) \right\}.
\]

given the independence of individuals between groups.

\[
E \left\{ \exp \left( \frac{i}{\sqrt{n}} \sum_{h,j} \theta_{h,j} W_{h,j} \right) \right\} = \prod_{j=1}^{k} E \left\{ \exp \left( \frac{i}{\sqrt{n}} \sum_{p=1}^{n_j} \int_0^T \Theta_j(s) \frac{dM_{j,p}(s)}{S_{P,j,p}(s)} \right) \right\},
\]

where \(\Theta_j(s) = \sum_{h=1}^{k} \theta_{h,j} (\delta_{hj} - a_h(s)) \mathbb{1}(y_u(s) > 0)\).

\(j\) being fixed, the variables \(\left( \int_0^T \Theta_j(s) \frac{dM_{j,p}(s)}{S_{P,j,p}(s)} \right)_{1 \leq p \leq n_j}\) are centered, independent identically distributed, with common variance

\[
E \left\{ \int_0^T \Theta_j^2(s) d\langle M_{j,1} \rangle (s) \right\} = E \left\{ \int_0^T \Theta_j^2(s) S_{C,j}(s) S_{E}(s) \left( \tilde{\lambda}_{P,j,1}(s) + \lambda_E(s) \right) ds \right\}.
\]

Hence, according to the central limit theorem, for all \(j \in [1;k]\)

\[
E \left\{ \exp \left( \frac{i}{\sqrt{n}} \sum_{p=1}^{n_j} \int_0^T \Theta_j(s) \frac{dM_{j,p}(s)}{S_{P,j,p}(s)} \right) \right\} \rightarrow \exp \left[ \frac{-\alpha_j}{2} \right] E \left\{ \int_0^T \Theta_j^2(s) S_{C,j}(s) S_{E}(s) \left( \tilde{\lambda}_{P,j,1}(s) + \lambda_E(s) \right) ds \right\}
\]
Therefore,

\[ E \left\{ \exp \left( \frac{i}{\sqrt{n}} \sum_{h,j} \theta_{h,j} W_{h,j} \right) \right\} \]

\[ \rightarrow_{n \to \infty} \exp \left[ \frac{1}{2} E \left\{ \sum_{j=1}^{k} \alpha_j \int_{0}^{T} \Theta_j^2(s) \frac{S_{C_j}(s) S_E(s)}{S_{P_j,1}(s)} \left( \hat{\lambda}_{P_j,1}(s) + \lambda_E(s) \right) ds \right\} \right] \]

\[ = \exp \left[ \frac{1}{2} \sum_{j=1}^{k} \sum_{h',j=1}^{k} \theta_{h',j} \theta_{h,j} \alpha_j E \left\{ \int_{0}^{T} \left( \delta_{h,j} - a_h(s) \right) \left( \delta_{h',j} - a_{h'}(s) \right) \times \right. \right. \]

\[ \left. \left. 1(y_w(s) > 0) \frac{S_{C_j}(s) S_E(s)}{S_{P_j,1}(s)} \left( \hat{\lambda}_{P_j,1}(s) + \lambda_E(s) \right) ds \right\} \right] \]

\[ = \exp \left\{ -\frac{1}{2} \sum_{h',j=1}^{k} \theta_{h',j} \theta_{h,j} S_{(h,j),(h')}^2 \right\}, \]

which proves (*).

Note that \[ \frac{1}{\sqrt{n}} (V_1, \ldots, V_k) = \frac{1}{\sqrt{n}} \left( \sum_{l=1}^{k} W_{1,l}, \ldots, \sum_{l=1}^{k} W_{k,l} \right). \]

Hence \[ \frac{1}{\sqrt{n}} (V_1, \ldots, V_k) \xrightarrow{d} N(0, \Sigma^2(T)), \] where \( \Sigma_{hj}(T) = \sum_{l,l'=1}^{k} S_{(hl),(jl')}^2 = \sum_{l=1}^{k} S_{(hl),(jl')}^2. \)

**Proof of Lemma 2.**

Let us denote

\[ R_{h,l}^T = \sum_{i=1}^{n_l} \int_{0}^{T} \left\{ 1(Y_w(s) > 0) \left( \delta_{hl} - \frac{Y_h(s)}{Y_w(s)} \right) - 1(y_w(s) > 0) (\delta_{hl} - a_h(s)) \right\} dM_{i,l}(s) \frac{S_{P_{i,l}}(s)}{S_{P_{i,l}}^2(s)}. \]

so that \( R_h^T = \sum_{l=1}^{k} R_{h,l}^T. \) We have

\[ E \left( \frac{1}{\sqrt{n}} R_{h,l}^T \right)^2 = E \left\{ \int_{0}^{T} f_n(s, \omega)^2 \left( \frac{1}{n} \sum_{i=1}^{n_l} Y_{i,l}(s) \frac{\hat{\lambda}_{P_{i,l}}(s) + \lambda_E(s)}{S_{P_{i,l}}^2(s)} \right) ds \right\} \]

where \( f_n : (s, \omega) \in (\mathbb{R} \times \Omega) \mapsto 1(Y_w(s) > 0) \left( \delta_{hl} - \frac{Y_h(s)}{Y_w(s)} \right) - 1(y_w(s) > 0) (\delta_{hl} - a_h(s)). \)

Then by Cauchy-Schwarz inequality

\[ E \left( \frac{1}{\sqrt{n}} R_{h,l}^T \right)^2 \leq \sqrt{E \left\{ \int_{0}^{T} f_n(s, \omega)^4 ds \right\}} \sqrt{E \left\{ \int_{0}^{T} \left( \frac{1}{n} \sum_{i=1}^{n_l} Y_{i,l}(s) \frac{\hat{\lambda}_{P_{i,l}}(s) + \lambda_E(s)}{S_{P_{i,l}}^2(s)} \right)^2 ds \right\}}. \]

By the law of large numbers, under the null hypothesis, \( f_n(s, \omega) \xrightarrow{n \to \infty} 0 \) (a.s.). Moreover, \( \forall(s, \omega) \in (\mathbb{R} \times \Omega), \ | f_n(s, \omega) | \leq 2. \) According to Lebesgue’s dominated convergence theorem,
we get

\[ E \left( \int_0^T f_n(s, \omega)^4 ds \right) \xrightarrow{n \to \infty} 0 \text{ under the null.} \]

On the other hand, since

\[
\left( \frac{1}{n} \sum_{i=1}^{n} Y_{l,i}(s) \frac{\tilde{\lambda}_{P,l,i}(s) + \lambda_E(s)}{S_{P,l,i}^2(s)} \right)^2 \leq 2 \frac{n}{n^2} \sum_{i=1}^{n} Y_{l,i}^2(s) \left( \tilde{\lambda}_{P,l,i}^2(s) + \lambda_E^2(s) \right)
\]

we have

\[
E \left\{ \int_0^T \left( \frac{1}{n} \sum_{i=1}^{n} Y_{l,i}(s) \frac{\tilde{\lambda}_{P,l,i}(s) + \lambda_E(s)}{S_{P,l,i}^2(s)} \right)^2 ds \right\}
\]

\[
\leq 2 \frac{n^2}{n^2} \int_0^T E \left\{ \frac{S_{P,l,i}^2(s) S_{P,l,i}^2(s)}{S_{P,l,i}^2(s)} \left( \tilde{\lambda}_{P,l,i}^2(s) + \lambda_E^2(s) \right) \right\} ds
\]

\[
\leq 2 E \left( \int_0^T \tilde{\lambda}_{P,l,i}(s) ds + \int_0^T \lambda_E^2(s) ds \right)
\]

\< \infty, \text{ according to assumptions (2) in the main document.}

We deduce from this that under the null hypothesis, \( \frac{1}{\sqrt{n}} R_{T}^{h,i} \xrightarrow{n \to \infty} 0, \forall (l, h) \in [1; k]^2 \), which ends the proof of Lemma 2.

Using (1), to prove that the asymptotic distribution of the test statistic is \( \chi^2_{k-1} \), it remains to prove

**Lemma 3:**

1. \( \frac{1}{n} \sigma^2_{h,j}(T) \xrightarrow{P} \Sigma^2_{h,j}(T). \)

2. \( \text{Ker}(\Sigma^2(T)) = \text{Vect} \{ \mathbf{I} \} \) and hence the matrix \( \Sigma^2_0(T) = (\Sigma^2_{h,j}(T))_{1 \leq h, j \leq k-1} \) is a symmetric positive definite matrix.

Point (2) of Lemma 3 ensures that we can delete the last row and the last column to use matrix \( \hat{\Sigma}_0^{2,w}(T) \) in formula (6) of the main document.
Proof of Lemma 3.

(1)

\[
\hat{\sigma}_{h,j}^2(T) = \left[ Z_h^w, Z_j^w \right](T)
\]

\[
= \sum_{l=1}^{k} \int_0^{T} 1(Y_{w}^h(s) > 0) \left( \delta_{hl} - \frac{Y_{h}^w(s)}{Y_{w}^h(s)} \right) \left( \delta_{jl} - \frac{Y_{j}^w(s)}{Y_{w}^j(s)} \right) \frac{dN_{l,i}(s)}{S_{P,l,i}^2(s)}
\]

=: N_T - Q_T

with

\[
N_T = \sum_{l=1}^{k} \int_0^{T} 1(y_{w}^h(s) > 0)(\delta_{hl} - a_h(s))(\delta_{jl} - a_j(s)) \frac{dN_{l,i}(s)}{S_{P,l,i}^2(s)}
\]

and

\[
Q_T = \sum_{l=1}^{k} \int_0^{T} \left\{ 1(Y_{w}^h(s) > 0) \left( \delta_{hl} - \frac{Y_{h}^w(s)}{Y_{w}^h(s)} \right) \left( \delta_{jl} - \frac{Y_{j}^w(s)}{Y_{w}^j(s)} \right) \\
- 1(y_{w}^h(s) > 0)(\delta_{hl} - a_h(s))(\delta_{jl} - a_j(s)) \right\} \times \sum_{i=1}^{n_l} dN_{l,i}(s) \frac{S_{P,l,i}^2(s)}{S_{P,l,i}^2(s)}
\]

Firstly, by the law of large numbers, we have for all \((h, j, l) \in [1; k]^3\)

\[
\frac{1}{n} \sum_{i=1}^{n_l} \int_0^{T} 1(y_{w}^h(s) > 0)(\delta_{hl} - a_h(s))(\delta_{jl} - a_j(s)) \frac{dN_{l,i}(s)}{S_{P,l,i}^2(s)} \xrightarrow{(a.s.)} S_{(h,l),(j,l)}^2(T).
\]

Thus \(\frac{1}{n} N_T \xrightarrow{n \to \infty} k \sum_{l=1}^{k} S_{(h,l),(j,l)}^2(T).

Secondly, we can show that \(\frac{1}{n} Q_T \xrightarrow{L^1} 0\). Indeed,

\[
\frac{1}{n} \mathbb{E}[|Q_T|] \leq \sum_{l=1}^{k} \frac{1}{n} \mathbb{E} \left\{ \sum_{i=1}^{n_l} \int_0^{T} \left| 1(Y_{w}^h(s) > 0) \left( \delta_{hl} - \frac{Y_{h}^w(s)}{Y_{w}^h(s)} \right) \left( \delta_{jl} - \frac{Y_{j}^w(s)}{Y_{w}^j(s)} \right) \right| Y_{l,i}(s) \frac{\lambda_{P,l,i}(s) + \lambda_E(s)}{S_{P,l,i}^2(s)} ds \right\}
\]

\[
\leq \sum_{l=1}^{k} \frac{n_l}{n} \sqrt{\mathbb{E}(A_T)} \times \sqrt{\mathbb{E}(B_T)}
\]
with
\[
A_T = \int_0^T \left[ 1 \left( Y^w(s) > 0 \right) \left( \delta_{hl} - \frac{Y^w_h(s)}{Y^w(s)} \right) \left( \delta_{jl} - \frac{Y^w_j(s)}{Y^w(s)} \right) \\
- 1 \left( y^w(s) > 0 \right) (\delta_{hl} - a_h(s))(\delta_{jl} - a_j(s)) \right]^2 ds
\]
\[
B_T = \int_0^T \frac{S_{C,l}(s)S_E(s)}{\tilde{S}^3_{P,l,1}(s)} \left( \frac{\lambda_{P,l,1}(s) + \lambda_E(s)}{\lambda_{P,l,1}(s)} \right)^2 ds
\]

We have \( A_T \xrightarrow{L^1} 0 \) and \( E(B_T) < \infty \) (see the proof of Lemma 2). So, we have shown that \( \frac{1}{n} \mathbb{E} \left[ Z^w_{hl}, Z^w_{j} \right] (T) \xrightarrow{P} \Sigma^2_{hj}(T) \).

(2) Let us denote by \( \mathbb{I} \) and \( \bar{u}(s) \) the vectors of \( \mathbb{R}^k \) given by

\[
\mathbb{I} := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad \bar{u}(s) := \begin{pmatrix} a_1(s) \\ \vdots \\ a_k(s) \end{pmatrix}.
\]

Then

\[
\Sigma^2(T) = E \left\{ \int_0^T (\text{Id} - \bar{u}(s)^t \mathbb{I}) D(s)^t (\text{Id} - \bar{u}(s)^t \mathbb{I}) ds \right\}
\]
\[
= \int_0^T A(s) D(s)^t A(s) ds,
\]

where

\[
A(s) = \text{Id} - \bar{u}(s)^t \mathbb{I}, \quad \text{and}
\]
\[
D(s) = \text{diag}(d_i(s))
\]
\[
= \text{diag} \left[ a_i E \left\{ 1 \left( y^w(s) > 0 \right) \frac{S_{C,l}(s)S_E(s)}{\tilde{S}^3_{P,l,1}(s)} \left( \frac{\lambda_{P,l,1}(s) + \lambda_E(s)}{\lambda_{P,l,1}(s)} \right) \right\} \right].
\]

Note that when \( \bar{v} \in \text{Vect}(\mathbb{I})^\perp \), \( A(s) \bar{v} = \bar{v} - \bar{u}(s) \langle \bar{v}, \mathbb{I} \rangle = \bar{v} \).

Hence \( \forall s \in [0, T], \text{Vect}(\mathbb{I})^\perp \subset \text{Im}(A(s)) \).

On the other hand,

\[
^t A(s) \mathbb{I} = \mathbb{I} - \mathbb{I} \langle \bar{u}(s), \mathbb{I} \rangle = \mathbb{I} - \mathbb{I} = \bar{0},
\]

since \( \langle \bar{u}(s), \mathbb{I} \rangle = \sum_{l=1}^k a_l(s) = 1 \).

Hence \( \text{Vect}(\mathbb{I}) \subset \text{Ker}(^t A(s)) = \text{Im}(A(s))^\perp \subset \text{Vect}(\mathbb{I}) \).
We thus get $\forall s \in [0, T], \text{Ker}(t^tA(s)) = \text{Vect}(I)$.

Now let $\vec{v} \in \text{Ker}(\Sigma^2(T))$, then

$$\langle \vec{v}, \Sigma^2(T)\vec{v} \rangle = 0$$

$$\iff \int_0^T \sum_{l=1}^k d_l(s) \left( t^t A(s) \vec{v} \right)_l^2 ds = 0$$

$$\implies ds \text{ a.e., } \forall l \in [1; k] \quad d_l(s) \left( t^t A(s) \vec{v} \right)_l^2 = 0.$$

Assume now that $\int_0^T d_l(s) ds > 0$. Then $\{s/d_l(s) > 0\}$ is a set of positive Lebesgue measure. On this set, we get $(t^t A(s) \vec{v})_l = 0$.

Therefore, there exists a set of positive measures $\mathcal{S}$ such that $t^t A(s) \vec{v} = 0, \forall s \in \mathcal{S}$

$$\iff \vec{v} \in \text{Ker}(t^tA(s)) = \text{Vect}(I), \forall s \in \mathcal{S}.$$

Hence, $\text{Ker}(\Sigma^2(T)) \subset \text{Vect}(I)$ and it is clear that $\Sigma^2(T)I = \int_0^T A(s)D(s)t^t A(s)I ds = 0,$

so that $\text{Ker}(\Sigma^2(T)) = \text{Vect}(I)$.

$\blacksquare$

**Web Table A**

[Table 1 about here.]

**Web Table B**

[Table 2 about here.]

**Web Table C**

[Table 3 about here.]
Table 1

*Comparison between two groups: percentage of rejection of the null hypothesis at the 5% level of significance with 2000 simulations of 1000 patients. The age distributions are scenario-specific: Scenario 1: 25% aged [40 – 64], 35% aged [65 – 74], and 40% aged [75 – 85]; Scenario 2: 30 ≤ age ≤ 40 (uniform); Scenario 3: 65 ≤ age ≤ 80 (uniform).*

<table>
<thead>
<tr>
<th>HR(^a)</th>
<th>Percentage of rejection (95% CI)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Proposed test</td>
</tr>
<tr>
<td><strong>Scenario 1(^b)</strong></td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>63.85 (61.74;65.96)</td>
</tr>
<tr>
<td>0.8</td>
<td>34.50 (32.42;36.58)</td>
</tr>
<tr>
<td>0.9</td>
<td>10.90 (9.53;12.27)</td>
</tr>
<tr>
<td>1</td>
<td>4.45 (3.55;5.35)</td>
</tr>
<tr>
<td>1.2</td>
<td>31.45 (29.42;33.48)</td>
</tr>
<tr>
<td>1.4</td>
<td>78.65 (76.85;80.45)</td>
</tr>
<tr>
<td>1.6</td>
<td>97.90 (97.27;98.53)</td>
</tr>
<tr>
<td><strong>Scenario 2(^b)</strong></td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>82.80 (81.15;84.45)</td>
</tr>
<tr>
<td>0.8</td>
<td>47.15 (44.96;49.34)</td>
</tr>
<tr>
<td>0.9</td>
<td>11.35 (9.96;12.74)</td>
</tr>
<tr>
<td>1</td>
<td>5.25 (4.27;6.23)</td>
</tr>
<tr>
<td>1.2</td>
<td>40.70 (38.55;42.85)</td>
</tr>
<tr>
<td>1.4</td>
<td>89.15 (87.79;90.51)</td>
</tr>
<tr>
<td>1.6</td>
<td>99.40 (99.06;99.74)</td>
</tr>
<tr>
<td><strong>Scenario 3(^b)</strong></td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>65.50 (63.42;67.58)</td>
</tr>
<tr>
<td>0.8</td>
<td>33.55 (31.48;35.62)</td>
</tr>
<tr>
<td>0.9</td>
<td>10.95 (9.58;12.32)</td>
</tr>
<tr>
<td>1</td>
<td>4.30 (3.41;5.19)</td>
</tr>
<tr>
<td>1.2</td>
<td>31.95 (29.91;33.99)</td>
</tr>
<tr>
<td>1.4</td>
<td>77.15 (75.31;78.99)</td>
</tr>
<tr>
<td>1.6</td>
<td>97.70 (97.04;98.36)</td>
</tr>
</tbody>
</table>

\(a\): Hazard Ratio of the level of \(G\) to the excess mortality used for data generation, where \(G\) is the covariate that represents the group.

\(b\): With unbalanced cases, which corresponds to cases where the groups are not similar in size with \(P(G = 0) = 1/4\) and \(P(G = 1) = 3/4\).
Comparison between three groups: percentage of rejection of the null hypothesis at the 5% level of significance with 2000 simulations of 1000 patients. The age distribution are scenario-specific: Scenario 2: 30 ≤ age ≤ 40 (uniform); Scenario 3: 65 ≤ age ≤ 80 (uniform).

<table>
<thead>
<tr>
<th>Scenario 2b</th>
<th>Proposed test</th>
<th>Usual log-rank test on hypothetical data</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 0.7)</td>
<td>84.45 (82.86;86.04)</td>
<td>84.60 (83.02;86.18)</td>
</tr>
<tr>
<td>(1, 1)</td>
<td>4.70 (3.77;5.63)</td>
<td>4.90 (3.95;5.85)</td>
</tr>
<tr>
<td>(1, 1.2)</td>
<td>34.60 (32.52;36.68)</td>
<td>36.30 (34.19;38.41)</td>
</tr>
<tr>
<td>(1, 1.4)</td>
<td>88.00 (86.58;89.42)</td>
<td>89.80 (88.47;91.13)</td>
</tr>
<tr>
<td>(1, 1.6)</td>
<td>99.55 (99.26;99.84)</td>
<td>99.70 (99.46;99.94)</td>
</tr>
<tr>
<td>(0.9, 1.2)</td>
<td>56.40 (54.23;58.57)</td>
<td>57.10 (54.93;59.27)</td>
</tr>
<tr>
<td>(0.8, 1.4)</td>
<td>99.35 (99.00;99.70)</td>
<td>99.50 (99.19;99.81)</td>
</tr>
<tr>
<td>(0.7, 1.6)</td>
<td>100 (99.81;100)</td>
<td>100 (99.81;100)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Scenario 3b</th>
<th>Proposed test</th>
<th>Usual log-rank test on hypothetical data</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 0.7)</td>
<td>69.60 (67.58;71.62)</td>
<td>83.15 (81.51;84.79)</td>
</tr>
<tr>
<td>(1, 1)</td>
<td>4.50 (3.59;5.41)</td>
<td>4.20 (3.32;5.08)</td>
</tr>
<tr>
<td>(1, 1.2)</td>
<td>26.45 (24.52;28.38)</td>
<td>33.95 (31.87;36.03)</td>
</tr>
<tr>
<td>(1, 1.4)</td>
<td>76.90 (75.05;78.75)</td>
<td>88.50 (87.10;89.90)</td>
</tr>
<tr>
<td>(1, 1.6)</td>
<td>96.75 (95.97;97.53)</td>
<td>99.30 (98.93;99.67)</td>
</tr>
<tr>
<td>(0.9, 1.2)</td>
<td>47.70 (45.51;49.89)</td>
<td>60.45 (58.31;62.59)</td>
</tr>
<tr>
<td>(0.8, 1.4)</td>
<td>97.20 (96.48;97.92)</td>
<td>99.40 (99.06;99.74)</td>
</tr>
<tr>
<td>(0.7, 1.6)</td>
<td>100 (99.81;100)</td>
<td>100 (99.81;100)</td>
</tr>
</tbody>
</table>

a: Hazard Ratio of the level of G to the excess mortality used for data generation, where G is the covariate that represents the group.
b: With balanced cases, which corresponds to the cases where the groups are similar in size with \( P(G = 0) = P(G = 1) = P(G = 2) \).
Table 3
Description of the SEER real dataset of patients diagnosed with colorectal cancer in 1998.

<table>
<thead>
<tr>
<th>Prognostic factors</th>
<th>Numbers</th>
<th>(%)(^a)</th>
<th>Deaths at 5 years</th>
<th>(%)(^b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Age</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>&lt; 70 (age1)</td>
<td>4332</td>
<td>(47.5)</td>
<td>1552</td>
<td>(35.8)</td>
</tr>
<tr>
<td>70 – 79 (age2)</td>
<td>2757</td>
<td>(30.2)</td>
<td>1320</td>
<td>(47.9)</td>
</tr>
<tr>
<td>⩾ 80 (age3)</td>
<td>2034</td>
<td>(22.3)</td>
<td>1349</td>
<td>(66.3)</td>
</tr>
<tr>
<td>Sex</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Man</td>
<td>4722</td>
<td>(51.8)</td>
<td>2137</td>
<td>(45.3)</td>
</tr>
<tr>
<td>Woman</td>
<td>4401</td>
<td>(48.2)</td>
<td>2084</td>
<td>(47.4)</td>
</tr>
<tr>
<td>Ethnicity</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Black</td>
<td>902</td>
<td>(9.9)</td>
<td>471</td>
<td>(52.2)</td>
</tr>
<tr>
<td>White</td>
<td>8211</td>
<td>(90.1)</td>
<td>3750</td>
<td>(45.6)</td>
</tr>
<tr>
<td>Tumor Stage at diagnosis</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Stage I</td>
<td>2132</td>
<td>(23.4)</td>
<td>501</td>
<td>(23.5)</td>
</tr>
<tr>
<td>Stage II</td>
<td>2880</td>
<td>(31.6)</td>
<td>1008</td>
<td>(35.0)</td>
</tr>
<tr>
<td>Stage III</td>
<td>2669</td>
<td>(29.2)</td>
<td>1385</td>
<td>(51.9)</td>
</tr>
<tr>
<td>Stage IV</td>
<td>1442</td>
<td>(15.8)</td>
<td>1327</td>
<td>(92.0)</td>
</tr>
<tr>
<td>Ethnicity × Tumor Stage at diagnosis</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>× Age class</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Black × Stage I × age1</td>
<td>98</td>
<td>(1.1)</td>
<td>13</td>
<td>(13.3)</td>
</tr>
<tr>
<td>Black × Stage I × age2</td>
<td>44</td>
<td>(0.5)</td>
<td>19</td>
<td>(43.2)</td>
</tr>
<tr>
<td>Black × Stage I × age3</td>
<td>22</td>
<td>(0.2)</td>
<td>13</td>
<td>(59.1)</td>
</tr>
<tr>
<td>Black × Stage II × age1</td>
<td>140</td>
<td>(1.5)</td>
<td>34</td>
<td>(24.3)</td>
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<tr>
<td>Black × Stage II × age2</td>
<td>90</td>
<td>(1.0)</td>
<td>38</td>
<td>(42.2)</td>
</tr>
<tr>
<td>Black × Stage II × age3</td>
<td>32</td>
<td>(0.4)</td>
<td>22</td>
<td>(68.8)</td>
</tr>
<tr>
<td>Black × Stage III × age1</td>
<td>168</td>
<td>(1.8)</td>
<td>81</td>
<td>(48.2)</td>
</tr>
<tr>
<td>Black × Stage III × age2</td>
<td>75</td>
<td>(0.8)</td>
<td>42</td>
<td>(56.0)</td>
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<tr>
<td>Black × Stage III × age3</td>
<td>41</td>
<td>(0.5)</td>
<td>31</td>
<td>(75.6)</td>
</tr>
<tr>
<td>Black × Stage IV × age1</td>
<td>124</td>
<td>(1.4)</td>
<td>111</td>
<td>(89.5)</td>
</tr>
<tr>
<td>Black × Stage IV × age2</td>
<td>48</td>
<td>(0.5)</td>
<td>47</td>
<td>(97.9)</td>
</tr>
<tr>
<td>Black × Stage IV × age3</td>
<td>20</td>
<td>(0.2)</td>
<td>20</td>
<td>(100)</td>
</tr>
<tr>
<td>White × Stage I × age1</td>
<td>917</td>
<td>(10.1)</td>
<td>78</td>
<td>(8.5)</td>
</tr>
<tr>
<td>White × Stage I × age2</td>
<td>636</td>
<td>(7.0)</td>
<td>166</td>
<td>(26.1)</td>
</tr>
<tr>
<td>White × Stage I × age3</td>
<td>415</td>
<td>(4.5)</td>
<td>212</td>
<td>(51.1)</td>
</tr>
<tr>
<td>White × Stage II × age1</td>
<td>1052</td>
<td>(11.5)</td>
<td>189</td>
<td>(18.0)</td>
</tr>
<tr>
<td>White × Stage II × age2</td>
<td>822</td>
<td>(9.0)</td>
<td>302</td>
<td>(36.7)</td>
</tr>
<tr>
<td>White × Stage II × age3</td>
<td>744</td>
<td>(8.2)</td>
<td>423</td>
<td>(56.9)</td>
</tr>
<tr>
<td>White × Stage III × age1</td>
<td>1181</td>
<td>(12.9)</td>
<td>462</td>
<td>(39.1)</td>
</tr>
<tr>
<td>White × Stage III × age2</td>
<td>677</td>
<td>(7.4)</td>
<td>369</td>
<td>(54.5)</td>
</tr>
<tr>
<td>White × Stage III × age3</td>
<td>527</td>
<td>(5.8)</td>
<td>400</td>
<td>(75.9)</td>
</tr>
<tr>
<td>White × Stage IV × age1</td>
<td>652</td>
<td>(7.1)</td>
<td>584</td>
<td>(89.6)</td>
</tr>
<tr>
<td>White × Stage IV × age2</td>
<td>365</td>
<td>(4.0)</td>
<td>337</td>
<td>(92.3)</td>
</tr>
<tr>
<td>White × Stage IV × age3</td>
<td>233</td>
<td>(2.6)</td>
<td>228</td>
<td>(97.9)</td>
</tr>
</tbody>
</table>

\(a\): Percentage of all 9123 patients;  
\(b\): Percentage of patients in a given category who died within the first 5 years after diagnosis.